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Analysis of the Euler–Poisson equations by methods of power geometry and normal form☆

A.D. Bruno
Moscow, Russia

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Abstract

New approaches and methods for studying non-linear problems are applied to the classical problem of the motion of a heavy rigid body about a fixed point, i.e., to the system of Euler–Poisson equations. All the asymptotic expansions of the solutions of the Kowalewski equations, to which the Euler–Poisson equations reduce when certain constraints are imposed on the parameters, are found using power geometry. They form 24 families. Then all the exact solutions of the Kowalewski equations of a specific class (which includes almost all the known exact solutions) are found on the basis of these expansions. Five new families of such solutions are found. Instead of the conventional technique of studying the global integrability of the Euler–Poisson equations, studying their local integrability near stationary and periodic solutions is proposed. Normal forms are used for this purpose. Sets of real stationary solutions, in the vicinity of which these equations are locally integrable, are discovered using them. Other real stationary solutions, in the vicinity of which the Euler–Poisson equations are locally non-integrable, are also found. This is established using the theory of resonant normal forms developed and computer calculations of the coefficients of a normal form.

1. Introduction

The problem of the motion of a heavy rigid body about a fixed point (a spinning top) was first examined by Euler.[1] These motions are now described by the autonomous system of six Euler–Poisson differential equations (see Ref. 2, in which the history of these equations is presented)

$$\begin{aligned} Ap' + (C-B)qr &= Mg(y_0\gamma_3 - z_0\gamma_2) \\ Bq' + (A-C)pr &= Mg(z_0\gamma_1 - x_0\gamma_3) \\ Cr' + (B-A)pq &= Mg(x_0\gamma_2 - y_0\gamma_1) \\ \gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' &= p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2 \end{aligned} \tag{1.1}$$

where the prime denotes differentiation with respect to the time t; A, B and C are the principal moments of inertia of the rigid body, which satisfy the triangle inequalities i.e.,

$$A > 0, \quad B > 0, \quad C > 0, \quad A + B \geq C, \quad A + C \geq B, \quad B + C \geq A \tag{1.2}$$

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E-mail address: brunoa@keldysg.ru.

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Mg is the weight of the body; x_0, y_0 and z_0 are the coordinates of the centre of gravity of the rigid body in a coordinate system associated with the body; p, q and r are the projections of the angular velocity vector onto the coordinate axes associated with the body, and γ_1, γ_2 and γ_3 are the direction cosines of a vertical in the coordinate system associated with the body. The system of Eq. (1.1) has three general first integrals (energy, momentum and geometric)

$$\begin{aligned} &I_1 \stackrel{\text{def}}{=} Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const} \\ &I_2 \stackrel{\text{def}}{=} Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const} \\ &I_3 \stackrel{\text{def}}{=} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \end{aligned} \tag{1.3}$$

System (1.1) can be integrated in quadratures if it has an additional (fourth) first integral.[2] An additional first integral I_4 has been found only in the following four cases:[2]

in the Euler–Poisson case

$$x_0 = y_0 = z_0 = 0 \text{ and } I_4 \stackrel{\text{def}}{=} A^2p^2 + B^2q^2 + C^2r^2 = \text{const} \tag{1.4}$$

in the Lagrange–Poisson case

$$B = C, \quad y_0 = z_0 = 0 \text{ and } I_4 \stackrel{\text{def}}{=} p = \text{const} \tag{1.5}$$

in the case of kinetic symmetry

$$A = B = C \text{ and } I_4 \stackrel{\text{def}}{=} x_0p + y_0q + z_0r = \text{const} \tag{1.6}$$

(this case is usually treated as a sub-case of the preceding one)
in the Kovalevskaya case

$$A = B = 2C, \quad y_0 = z_0 = 0 \text{ and } \tilde{I}_4 \stackrel{\text{def}}{=} (p^2 - q^2 - c\gamma_1)^2 + (2pq - c\gamma_2)^2 = \text{const} \tag{1.7}$$

where $c = Mgx_0/C$.

In Sections 2 and 3, system (1.1) is considered in the case when

$$B \neq C, \quad Mg = 1, \quad x_0 \neq 0, \quad y_0 = z_0 = 0 \tag{1.8}$$

In this case, Kowalewski[3] reduced the system (1.1) to the non-autonomous system of two equations with the independent variable p. In Section 2, all 24 families of asymptotic expansions of the solutions of the Kowalewski system as $p \to 0$ and $p \to \infty$ are obtained for all values of the parameters using power geometry. Several families of asymptotic expansions of the solutions of system (1.1) are obtained from them. In Section 3, all the non-empty intersections of the families of expansions as $p \to 0$ with the families as $p \to \infty$, i.e., all the solutions in the form of finite sums of powers of the independent variable p, are found for the Kowalewski equations. They correspond to exact solutions of the system of Eq. (1.1). Seven of them were previously found by Steklov, Goryachev, Chaplygin, Kowalewski, Appelratt and Gorr. All five families of new exact solutions are complex.

In Sections 4 and 5, we examine the case when

$$A = B = 1, \quad Mgx_0 = 1, \quad y_0 = z_0 = 0 \tag{1.9}$$

in which system (1.1) contains only one parameter: $C \in (0, 2]$. It has four families of stationary solutions (points). In Section 4, 44 sets of complex stationary solutions, in the vicinity of which system (1.1) is locally integrable, are isolated from these families using a normal form; 10 of these sets are real. Such non-empty real sets exist for all $C \in (0, 2]$. In addition, the stationary points at infinity (power-law asymptotic forms of the solutions), in the vicinity of which system (1.1) is locally integrable, are indicated. The existence of periodic solutions, in the vicinity of which system (1.1) is locally integrable, is discussed.

In Section 5, one-parameter families of stationary solutions, in the vicinity of which there is no additional (formal) first integral, i.e., system (1.1) is locally non-integrable, are indicated using a resonant normal form. Such stationary solutions exist for all $C \in (0, 2]$ except for two classical cases of global integrability. Lagrange–Poisson ($C = 1$) and Kovalevskaya ($C = 1/2$), which occur only in the case (1.9).

2. Asymptotic expansions of solutions of the Kowalewski equations

2.1. Power geometry[4]

Let $x_0 \stackrel{\text{def}}{=} t$ be independent, and let $x_1, \ldots, x_n$ be dependent variables, where $x_i \in \mathbb{C}$. We set $X = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$. The *differential monomial* $a(X)$ is the product of the ordinary monomial

$$cx_0^{m_0} x_1^{m_1} \ldots x_n^{m_n} \stackrel{\text{def}}{=} cX^M \tag{2.1}$$

where $c = \text{const} \in \mathbb{C}$ and $M = (x_0, m_1, \ldots, m_n) \in \mathbb{R}^{n+1}$, and a finite number of derivatives of the form

$$d^l x_i / dx_0^l \stackrel{\text{def}}{=} d^l x_i / dt^l, \quad l \in \mathbb{N}, \quad i > 0 \tag{2.2}$$

The sum of the differential monomials

$$f(X) = \sum a_i(X) \tag{2.3}$$

is called a *differential sum*.

Let the system of ordinary differential equations

$$f_i(X) = 0, \quad i = 1, \ldots, n \tag{2.4}$$

where the $f_i(X)$ are differential sums, be assigned. Let $t \to 0$ or $t \to \infty$, and let the solution of system (2.4) have the form

$$x_i = t^{r_i}(b_i + \sum b_{is} t^s), \quad s \in \mathbf{K}, \quad b_i \neq 0, \quad i = 1, \ldots, n \tag{2.5}$$

where the coefficients $b_i = \text{const} \in \mathbb{C}$ and the exponents $r_i, s \in \mathbb{C}$. Then the expression

$$x_i = b_i t^{r_i}, \quad b_i \neq 0, \quad i = 1, \ldots, n \tag{2.6}$$

is a *power-law asymptotic* form of solution (2.5).

All the power-law asymptotic forms (2.6) of the solutions (2.5) of system (2.4) can be found in the following way.[4–7] Each differential monomial $a(X)$ is mapped to its (vector) exponent $Q(a) = (q_0, q_1, \ldots, q_n) \in \mathbb{R}^{n+1}$ according to the following rules. For a monomial of the form (2.1) we take $Q(cX^M) = M$, i.e., $Q(ct^{m_0} x_1^{m_1} \ldots x_n^{m_n}) = (m_0, m_1, \ldots, m_n)$; for a derivative of the form (2.2) we take $Q(d^l x_i / dt^l) = -lE_0 + E_i$, where E_i is the i-th unit vector in $\mathbb{R}^{n+1}$; when differential monomials are multiplied, their power exponents are summed as vectors: $Q(a_1 a_2) = Q(a_1) + Q(a_2)$. The set $\mathbf{S}(f)$ of the power exponents $Q(a_i)$ of all the differential monomials $a_i(X)$ included in the differential sum (2.3) is called the *support of the sum* $f(X)$. Obviously, $\mathbf{S}(f) \in \mathbb{R}^{n+1}$. The closure of the convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(X)$. The boundary $\partial\Gamma(f)$ of the polyhedron $\Gamma(f)$ consists of the generalized *faces* $\Gamma_j^{(d)}$, where the superscript indicates the dimension of the face and the subscript indicates its number. Each face $\Gamma_j^{(d)}$ has a corresponds to a *truncated sum*

$$\hat{f}_j^{(d)}(X) = \sum a_i(X) \quad \text{for} \quad Q(a_i) \in \Gamma_j^{(d)}$$

Let us now consider system (2.4). Each equation has its own support $\mathbf{S}_i \stackrel{\text{def}}{=} \mathbf{S}(f_i)$, its own polyhedron $\Gamma_i \stackrel{\text{def}}{=} \Gamma(f_i)$ with the faces $\Gamma_{ij}^{(d)}$ and the truncated equations $\hat{f}_{ij_i}^{(d_i)}(X) = 0$. The asymptotic forms (2.6) are found as solutions of the corresponding truncated systems of equations

$$\hat{f}_{ij_i}^{(d_i)}(X) = 0, \quad i = 1, \ldots, n$$

Power geometry enables us to find not only the asymptotic forms (2.6) of the solutions (2.5), but also expansions (2.5) themselves.[6,8,9] The coefficients b_{is} can be functions of $\ln t$, $\ln\ln t$, etc.

These asymptotic forms and expansions were calculated for the Kowalewski system (2.4). The results are presented below.

2.2. The Kowalewski equations

For the case (1.8), Kowalewski[3] proposed treating p as the independent variable, introduced the new dependent variables

$$\sigma = (B-C)q^2/A, \quad \tau = (B-C)r^2/A \tag{2.7}$$

and obtained the system of equations in the new variables

$$\begin{aligned} f_1 &\overset{\text{def}}{=} \ddot{\sigma}\tau + \dot{\sigma}\dot{\tau}/2 + a_1 + a_2\sigma + a_3\dot{\tau}p + a_4\tau + a_5p^2 = 0 \\ f_2 &\overset{\text{def}}{=} \sigma\ddot{\tau} + \dot{\sigma}\dot{\tau}/2 + b_1 + b_2\dot{\sigma}p + b_3\sigma + b_4\tau + b_5p^2 = 0 \end{aligned} \tag{2.8}$$

where the dot denotes differentiation with respect to p. Here

$$\begin{aligned} q^2 &= A\sigma/(B-C), \quad r^2 = A\tau/(B-C) \\ \gamma_1 &= [h - A(B\sigma + C\tau)/(B-C) - Ap^2]/(2x_0) \\ \gamma_2 &= -C[\dot{\tau} - 2(A-B)p/C]q/(2x_0) \\ \gamma_3 &= B[\dot{\sigma} - 2(C-A)p/B]r/(2x_0) \\ t &= \int\frac{dp}{\sqrt{\sigma\tau}} \end{aligned} \tag{2.9}$$

This change of coordinates uses the first of the integrals (1.3) and transforms the other two integrals to the respective forms

$$\begin{aligned} f_3 &\overset{\text{def}}{=} \dot{\sigma}\tau - \sigma\dot{\tau} + c_1 + c_2p + c_3\sigma p + c_4\tau p + c_5p^3 = 0 \\ f_4 &\overset{\text{def}}{=} d_1(\dot{\sigma})^2\tau + \sigma(\dot{\tau})^2 + d_2 + d_3\sigma + d_4\tau + d_5\sigma^2 + d_6\dot{\sigma}\tau p + \\ &+ d_7\sigma\dot{\tau}p + d_8\sigma\tau + d_9\tau^2 + d_{10}p^2 + d_{11}\sigma p^2 + d_{12}\tau p^2 + d_{13}p^4 = 0 \end{aligned} \tag{2.10}$$

We introduce the new parameters

$$x = A/C, \quad y = B/C, \quad z = h/C, \quad \lambda = l/C, \quad \xi = x_0/C \tag{2.11}$$

Then, for a_i, b_i in (2.8) and c_i, d_i in (2.10), we obtain

$$a_1 = -z/y, \quad a_2 = x/(y-1), \quad a_3 = (x-2)/y$$

$$a_4 = (2xy - x - 2y + 2)/(y(y-1)), \quad a_5 = (3x - 2y)/y$$

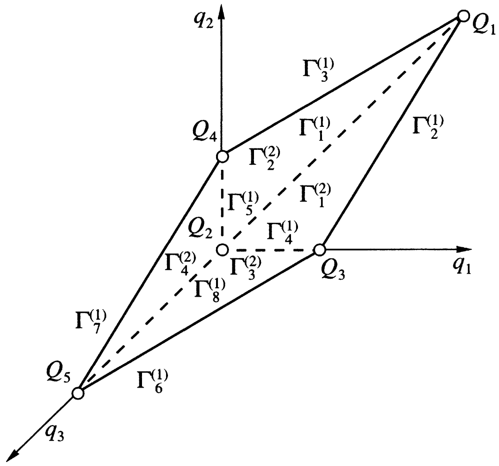

Fig. 1.

$$
\begin{aligned}
&b_1 = -z, \quad b_2 = -x+2y, \quad b_3 = (2y^2 - xy + 2x - 2y)/(y-1)\\
&b_4 = x/(y-1), \quad b_5 = 3x-2\\
&c_1 = -2(y-1)\lambda\xi/(xy), \quad c_2 = z(y-1)/y, \quad c_3 = x-2y\\
&c_4 = (x-2)/y, \quad c_5 = -x(y-1)/y\\
&d_1 = y^2, \quad d_2 = (z^2-4\xi^2)(y-1)/x, \quad d_3 = -2yz, \quad d_4 = -2z\\
&d_5 = xy^2/(y-1), \quad d_6 = 4(x-1)y, \quad d_7 = -4(x-y)\\
&d_8 = 2xy/(y-1), \quad d_9 = x/(y-1), \quad d_{10} = -2(y-1)z\\
&d_{11} = 2(2x^2+2y^2-3xy), \quad d_{12} = 2(2x^2-3x+2), \quad d_{13} = x(y-1)
\end{aligned}
\tag{2.12}
$$

where x and y satisfy the inequalities

$$x+y \geq 1, \quad x-y \geq -1, \quad y-x \geq -1, \quad y \neq 0, \quad y \neq 1 \tag{2.13}$$

Let **D** denote the corresponding set of points (x, y). In the x, y plane the set **D** is the half-strip bounded by the straight lines $y = x \pm 1$ and the segment $x + y = 1$ with excluded segment $y = 1$. The set **D** is located entirely in the first quadrant.

System (2.8) and integrals (2.10) possess the symmetry

$$(\sigma, \tau, p, x, y, z, \lambda, \xi) = (-\bar{\tau}, -\bar{\sigma}, \bar{p}, \bar{x}/y, 1/\bar{y}, \bar{z}/\bar{y}, \bar{\lambda}/\bar{y}, \bar{\xi}/\bar{y}) \tag{2.14}$$

In the generic case for system (2.8), the polyhedrons $\Gamma(f_1)$ and $\Gamma(f_2)$ are identical and are represented in Fig. 1, which shows the notation of the vertices $Q_1, \ldots, Q_5$, the edges $\Gamma_1^{(1)}, \ldots, \Gamma_8^{(1)}$ and the faces $\Gamma_1^{(2)}, \ldots, \Gamma_5^{(2)}$ of this polyhedron. The polyhedron $\Gamma(f_1)$ is a tetragonal pyramid with vertices $Q_1 = (1, 1, -2)$, $Q_2 = (0, 0, 0)$, $Q_3 = (1, 0, 0)$, $Q_4 = (0, 1, 0)$ and $Q_5 = (0, 0, 2)$. The base $\Gamma_5^{(2)}$ of this pyramid is spanned onto the vertices Q_1, Q_3, Q_4 and Q_5 and is on top in Fig. 1. The edges $\Gamma_4^{(1)}$, $\Gamma_5^{(1)}$ and $\Gamma_8^{(1)}$ are located on the q_1, q_2 and q_3 axes, respectively, and connect the vertices Q_3, Q_4 and Q_5 of the base $\Gamma_5^{(2)}$ to the vertex Q_2 of the pyramid. The edge $\Gamma_1^{(1)}$ connects the vertices Q_1 and Q_2. The lateral faces $\Gamma_3^{(2)}$ and $\Gamma_4^{(2)}$ of the pyramid lie in the (q_1, q_3) and (q_2, q_3) coordinate planes, respectively.

The following have been found for the solutions $\sigma(p)$ and $\tau(p)$ of the Kowalewski system (2.8) as $p \to 0$ and $p \to \infty$

a) all the power-law asymptotic forms[10–19]

$$\sigma = \sigma_0 p^{\alpha}, \quad \tau = \tau_0 p^{\beta}; \quad \sigma_0, \tau_0, \alpha, \beta \in \mathbb{C}, \quad \sigma_0, \tau_0 = \text{const} \neq 0 \tag{2.15}$$

b) all the power expansions[10–19] of the form

$$\sigma = \sigma_0 p^{\alpha} + \sum_s \sigma_s p^{\alpha+s}, \quad \tau = \tau_0 p^{\beta} + \sum_s \tau_s p^{\beta+s} \tag{2.16}$$

$$\alpha, \beta, s, \sigma_s, \tau_s \in \mathbb{C}, \quad \sigma_0, \tau_0, \sigma_s, \tau_s = \text{const}, \quad \sigma_0, \tau_0 \neq 0$$

Re $s > 0$ and $\omega = -1$ if $p \to 0$, and Re $s < 0$ and $\omega = 1$ if $p \to \infty$;

c) all the power logarithmic expansions of the form (2.16), where $\sigma_0, \tau_0 = \text{const} \neq 0$, and σ_s and τ_s are polynomials in $\ln p$;[6,14,18]
d) the exponentially small additions to the power and power logarithmic expansions;[9]
e) all the non-power-law asymptotic forms[5,14,18] of the form (2.15), where σ_0 and τ_0 are functions of $\ln p$ and $\ln\ln p$;
f) complicated expansions of the form (2.16), where σ_0, τ_0, σ_s and τ_s are series in decreasing powers of $\ln p$.[20]

These results are systematically presented below in Subsections 2.3–2.7.

2.3. Power-law asymptotic forms

All 24 families $F_1 - F_{24}$ of the power-law asymptotic forms (2.15) were found in Refs. 10–19. It was shown in Ref. 19 that they exhaust all the power-law asymptotic forms for all values of the parameters. The data for them are presented in Table 1. The first column contains the index k of the family F_k, the second column contains the index $\bar{k}$ of the symmetric family $F_{\bar{k}}$ obtained according to (2.14) ($F_{\bar{k}} \overset{\text{def}}{=} \overline{F_k}$), the third column contains the values of α, β and ω for the family F_k, the fourth, fifth and sixth columns contain the values of σ_0 and τ_0 if they are defined and the principal constraints in the domain of definition of the family F_k, and the last column indicates the vertex Q_j, the edge $\Gamma_j^{(1)}$ or the face $\Gamma_j^{(2)}$ of the polyhedron $\Gamma(f_1)$ that corresponds to the family F_k. Here $\varphi_0 = \sqrt{16xy^2 - 8x^2y + 9x^2 - 16xy}$. Fig. 2 shows the regions $\mathbf{F}_1$, $\mathbf{F}_1'$, $\mathbf{F}_0$ and $\mathbf{F}_2$ and their boundary curves $\alpha = -1$ and $\alpha = 4$ in the set $\mathbf{D}$, and Fig. 3 shows

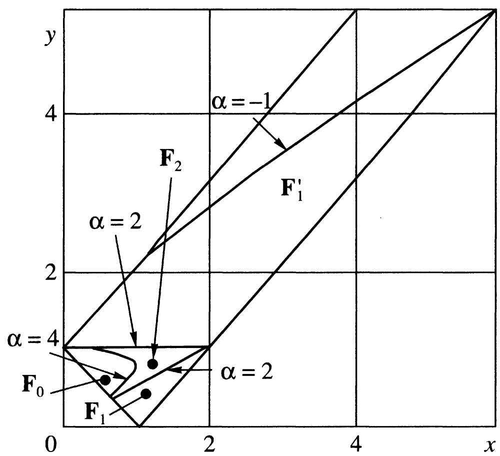

Fig. 2.

Table 1

k	$\bar{k}$	α, β, ω	σ_0	τ_0	Constraints	Face
1	2	$0, 1, -1$			$\lambda \neq 0$	Q_1
3	3	$0, 1, -1$				Q_1
4	4	$\frac{2}{3}, \frac{2}{3}, -1$			$\lambda = 0$	Q_1
5	6	$-1, 2, -1$		$\frac{x}{y-1}$		$\Gamma_2^{(1)}$
7	8	$\frac{x-\varphi_0}{2(x-2y)(y-1)}, 2, -1$		$\frac{x-2y}{2-\alpha}$	$(x, y) \in \mathbf{F}_1 \cup \mathbf{F}'_1$	$\Gamma_2^{(1)}$
9	10	$\frac{x-\varphi_0}{2(x-2y)(y-1)}, 2, 1$		$\frac{x-2y}{2-\alpha}$	$(x, y) \in \mathbf{F}_2 \cup \mathbf{F}_0$	$\Gamma_2^{(1)}$
11	12	$2, \frac{2}{3}, 1$	$\frac{y}{y-1}$		$x = y$	$\Gamma_6^{(1)}$
13	14	$2, \frac{y}{y-1}, 1$	$\frac{y}{y-1}$		$x = y > 2, \lambda \neq 0$	$\Gamma_6^{(1)}$
15	16	$2, \frac{y}{y-1}, 1$	$\frac{y}{y-1}$		$x = y > 2$	$\Gamma_6^{(1)}$
17	18	$2, \frac{y}{y-1}, 1$	$\frac{y}{y-1}$		$x = y > 2, \lambda = 0$	$\Gamma_6^{(1)}$
19	19	$2, 2, 1$	$\frac{x-1}{x-2y}$	$\frac{y-x}{x-2}$	$x \neq 1, 2, y, 2y$	$\Gamma_5^{(2)}$
20	21	$2, 2, 1$	$-\frac{1}{2}$		$x = y = 2$	$\Gamma_5^{(2)}$
22	22	$2, 2, 1$	$\frac{1-x}{y}$	$x - y$	$x \neq 1, y$	$\Gamma_5^{(2)}$
23	24	$\alpha \in (0, 1), 2-\alpha, -1$		$\frac{4\xi}{\sigma_0(\alpha-2)^2}$	$y = \frac{(\alpha-2)^2}{\alpha^2} > 1$ $\lambda = 0, z^2 = 4\xi^2$	$\Gamma_1^{(1)}$

the portions of the straight lines $x=1$, $x=2$, $x=y$ and $x=2y$, on which some of the families F_k are defined or are not defined, and the two points $x=y=2$ and $x=1$, $y=1/2$, at which the families F_{20} and F_2 are defined. In the set $\mathbf{D}$, the domains of definition of the families $F_1 - F_{10}$, F_{19} and $F_{22} - F_{24}$ are two-dimensional, the domains of definition of the families $F_{11} - F_{18}$ are one-dimensional, and the domains of definition of the families F_{20} and F_{21} consist of one point. It can be seen from Table 1 that the exponents α and β are complex numbers with non-zero real parts (Fig. 2) only for F_9 when $(x, y) \in \mathbf{F}_0$ and for F_{10} when $(x, y) \in \overline{\mathbf{F}_0}$. In all other cases, they are real.

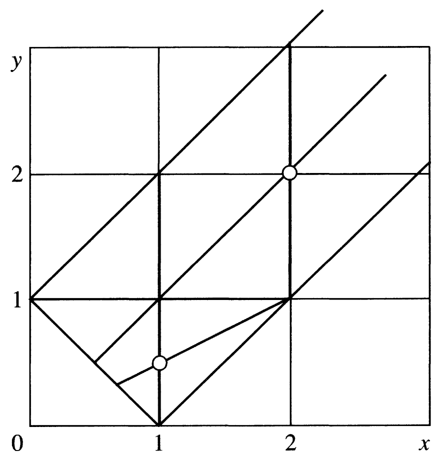

Fig. 3.

2.4. Power expansions

Each power-law asymptotic form

$$\sigma = \sigma_0 p^{\alpha}, \quad \tau = \tau_0 p^{\beta}, \quad \sigma_0, \tau_0 = \text{const} \neq 0 \tag{2.17}$$

of the solutions of system (2.8) has four eigenvalues $s_1, \ldots, s_4$. When the corresponding power expansions

$$\sigma = \sigma_0 p^{\alpha} + \sum \sigma_s p^{\alpha+s}, \quad \tau = \tau_0 p^{\beta} + \sum \tau_s p^{\beta+s}, \quad \sigma_s, \tau_s = \text{const} \tag{2.18}$$

of the solutions of system (2.8) are constructed, the coefficients σ_s and τ_s are successively determined for increasing or decreasing values of $|\mathrm{Re}\, s|$ from a system of linear equations. If the number s is not an eigenvalue, the matrix of the system is non-degenerate, and the coefficients σ_s and τ_s are uniquely defined. If s is an eigenvalue, the matrix of the system is degenerate, and it has a solution only when the compatibility condition holds. If this condition holds, a one-parameter family of the coefficients σ_s and τ_s exists. An eigenvalue s is called *critical* if $\omega\, \mathrm{Re}\, s < 0$. We recall that $\omega = -1$ if $p \to 0$ (in this case, $\mathrm{Re}\, s > 0$ in expansions (2.18)) and $\omega = 1$ if $p \to \infty$ (in this case, $\mathrm{Re}\, s < 0$). The critical eigenvalue s_i is called *dangerous* if there is a non-trivial compatibility condition for $s = s_i$. The two eigenvalues s_3 and s_4 correspond to the integrals f_3 and f_4 from relations (2.10) in the sense that substitution of expansions (2.18) into the integral f_i gives an expansion in powers of p in which σ_{s_i} and τ_{s_i} are first encountered with the zero power of p, as is the constant of the integral f_i ($i = 3, 4$). Therefore, the eigenvalues s_3 and s_4 are always non-dangerous. We order the remaining eigenvalues s_1 and s_2 as follows: $\mathrm{Re}\, s_1 \leq \mathrm{Re}\, s_2$. It turned out that only one of them can be dangerous, and this occurs in comparatively rare cases.

The eigenvalues $s_1, \ldots, s_4$ were calculated for the 24 families F_k of the power-law asymptotic forms (2.17), and the dangerous eigenvalues were identified. The family of the power expansions (2.18) that corresponds to the family F_k of the power-law asymptotes (2.17) is denoted by H_k. The results of these calculations are presented in Table 2. The first column contains the index k of the family, which is followed by the eigenvalues $s_1, \ldots, s_4$, the dangerous eigenvalues and the number of arbitrary coefficients in expansions (2.18). This table was compiled as a continuation of Table 1.

Table 2

k	s_1	s_2	s_3	s_4	Dangerous eigenvalues	Number of arbitrary constants
1	-1	0	0	$\frac{1}{2}$		3
3	0	0	1	1		4
4	-1	0	$-\frac{1}{3}$	0		2
5	$-\frac{5+\varphi_1}{4}$	$\frac{\varphi_1-5}{4}$	0	2	$s_2 - n$	2, 3
7	$-2+\frac{\alpha}{2}$	0	$-\alpha-1$	-2α		3
9	$-2+\frac{\alpha}{2}$	0	$-\alpha-1$	-2α	$s_1 = n(2-\alpha)$	3, 4
11	$\frac{1+\varphi_3}{6}$	$\frac{\varphi_3-1}{6}$	$-\frac{1}{3}$	0	$s_1 = -\frac{n+2}{3}$	2, 3
13	0	$\frac{3}{2}\beta-1$	0	$1-\beta$		3
15	0	0	$1-\frac{3}{2}\beta$	$2-\frac{5}{2}\beta$		4
17	0	$\frac{5}{2}\beta-2$	$\beta-1$	0		2
19	$\frac{1+\varphi_5}{2}$	$\frac{\varphi_5-1}{2}$	-3	-4	$s_1 = -\frac{1}{n+2}, s_1 = s_2 = \frac{1}{2}$	2, 3, 4
20	-1	0	-3	-4		4
22	-2	-1	-3	-3	$s_1 = 2s_2 = -2$	4
23	$\alpha-1$	0	-1	$-\alpha$		1

Here

$$\varphi_1 = \sqrt{(16y^2 - 8xy + 9x - 16y)/x}$$

$$\varphi_3 = \sqrt{(13y + 23)/(y - 1)}$$

$$\varphi_5 = \sqrt{(4x^2 - 8xy - 8x + 17y)/y}$$

and n is any natural number. It can be seen from Table 2 that the dangerous eigenvalues for the families H_5, H_6, H_9, H_{10} and H_{19} are encountered on curves in the set **D**. The dangerous curves with $n = 1, 2, 3, 4, 5$ for the families H_5, H_9 and H_{19} are shown in Fig. 4. The critical eigenvalues s_1 and s_2 for the family H_{19} are complex in the regions $\mathbf{H}_0$ and $\overline{\mathbf{H}_0}$, which lie below the lowermost curve and above the uppermost curve corresponding to $s = -1/2$. The dangerous eigenvalues for the family H_{11} form a denumerable set of points on the straight line $x = y$. In particular, this set includes the point $x = y = 2$, at which the additional Kovalevskaya first integral (1.7) exists. At this point, $s_1 = -4/3$ and $n = 2$. However, at this point, the compatibility condition for H_{11} holds. In addition, for $y < 1$, the expansions (2.18) of the family H_{11} have complex exponents because s_1 and s_2 are complex and Re $s_1 < 0$. Finally, for F_{22} there are no corresponding power expansions because the compatibility condition does not hold in its domain of definition (i.e., when $x \neq y$ and $x \neq 1$). Therefore, the family H_{22} is absent. According to the theory in Ref. 8, expansions (2.18) can diverge only for $H_{11} - H_{18}$, and they converge for the remaining families.

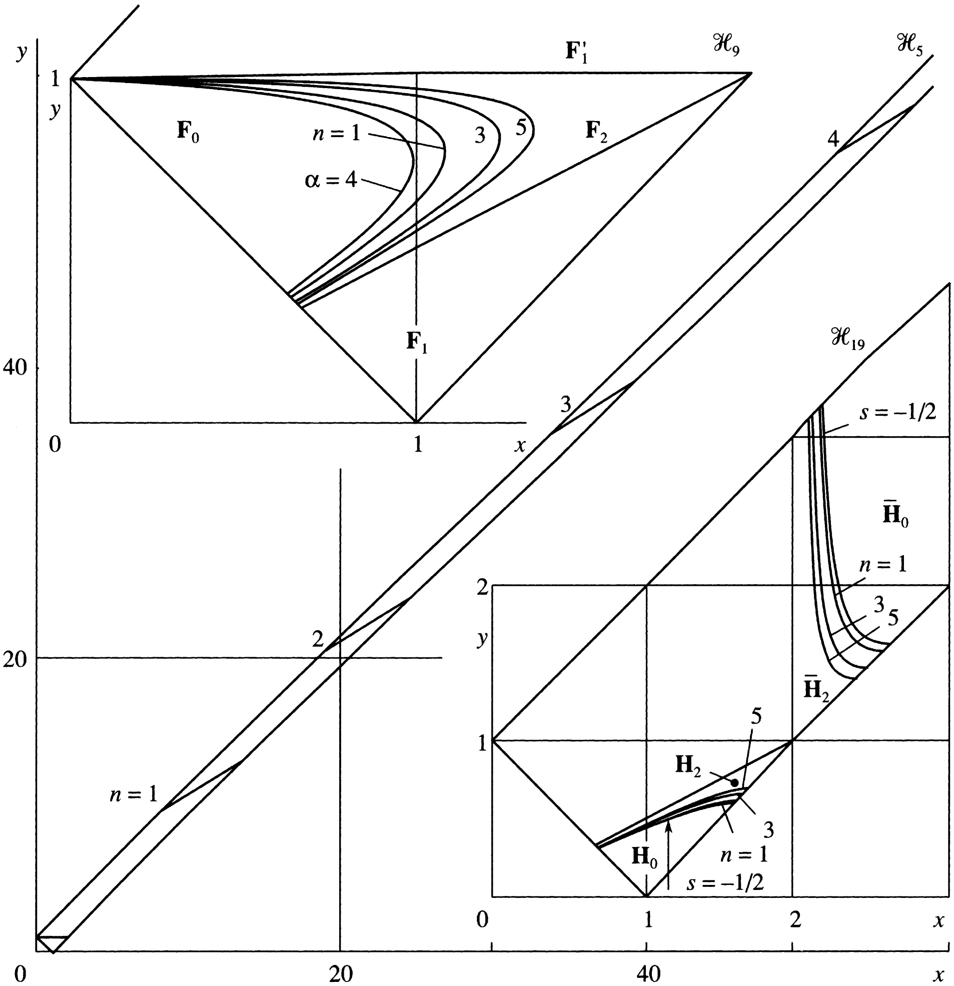

Fig. 4.

2.5. Power logarithmic expansions

All the power logarithmic expansions

$$\sigma = \sigma_0 p^{\alpha} + \sum \sigma_s(\ln p) p^{\alpha+s}, \quad \tau = \tau_0 p^{\beta} + \sum \tau_s(\ln p) p^{\beta+s} \tag{2.19}$$

of the solutions of system of Eq. (2.8), where σ_s and τ_s are polynomials in $\ln p$, were also found for the power-law asymptotic forms (2.15) (Ref. 6, Section 3.3). Such expansions appear only in cases in which the compatibility condition does not hold. According to Table 2, they exist for the seven families F_5, F_6, F_9, F_{10}, F_{11}, F_{12} and F_{19} on the dangerous curves and points in the set **D**. According to the results previously obtained (Ref. 6, Lemma 2.2 and Example 2.2), *one-parameter logarithms* are obtained, i.e., the coefficients of $\ln p$ in expansions (2.19) depend on one arbitrary constant, in all these cases. In addition, three cases of resonant critical eigenvalues s_i exist.

Case 1. For the families F_5 and F_6 when $s_2 = s_4 = 2$.
Case 2. For the families F_{19} when $s_1 = s_2 = -1/2$.
Case 3. For the families F_{22} over the entire domain of definition, i.e., in the region where $x \neq y$ and $x \neq 1$, when $s_1 = 2s_2 = 2$.

In these cases, *two-parameter logarithms* are obtained (Ref. 6, Lemma 2.2 and Example 2.3), i.e., the coefficients of $\ln p$ in expansions (2.19) depend on two arbitrary constants. In Cases 1 and 2, the compatibility condition holds, and the logarithms first appear in the first power. In Case 3, the compatibility condition does not hold, and the logarithms first appear in the second power. We will consider these cases separately.

Case 1. This case was described for the family F_5 (Ref. 11, Section 6.2) as Case 6 and was also discussed further (Ref. 13, Section 16.1). In the set **D**, it is realized on the curve

$$x = 2y(y-1)/(y+20) \tag{2.20}$$

(Fig. 4, the family H_5, $n = 2$). The expansions of the solutions have the form

$$\sigma = \sigma_0 p^{-1} + \sum_{s=1}^{\infty} \sigma_s p^{s-1}, \quad \tau = \tau_0 p^2 + \sum_{s=1}^{\infty} \tau_s p^{s+2}$$

where σ_0 is an arbitrary constant and $\tau_0 = x/(1-y)$. The equality (Ref. 6, Section 3.4)

$$\sigma_1\tau_1 = -z^2(y-9)(y-20)(y+20)/(11^2\sigma_0 y^3)$$

holds on curve (2.20), and the compatibility condition holds for $s = 2$. We seek σ_2 and τ_2 in the form

$$\sigma_2 = b_{10} + b_{11}\ln p, \quad \tau_2 = b_{20} + b_{21}\ln p \tag{2.21}$$

For the four coefficients b_{ij} we obtain the system of two equations

$$\begin{aligned} &2\psi b_{11} + 10\sigma_0 b_{21} = 0 \\ &2\psi b_{10} + 10\sigma_0 b_{20} + \psi b_{11} + (13\sigma_0/2)b_{21} + 6\sigma_1\tau_1 = 0 \end{aligned} \tag{2.22}$$

Here $\psi = 20y/(y+20)$. We take b_{10} and b_{11} as the arbitrary parameters, and then b_{20} and b_{21} are uniquely determined from the system of linear Eq. (2.22). If $b_{11} = 0$, then $b_{21} = 0$ according to the first equation in (2.22), and the coefficients σ_2 and τ_2 do not contain logarithms and depend on the single parameter $\sigma_2 = b_{10}$. Then the power expansion previously written down (Ref. 11, expansions (6.2.10) and (6.2.11)) is obtained.

The family F_6 is symmetric to the family F_5 according to (2.14). Therefore, the structure of the presence of $\ln p$ in expansions (2.19) for it is similar to the case in question.

Case 2 *(Ref. 12, Sections 8.3 and 8.7).* In the set **D**, this case is realized on the curve

$$y = 4x(x-2)/(8x-17) \tag{2.23}$$

(Fig. 4, the family H_{19}, $s=-1/2$) for $x\neq 1$, $x\neq y$. The expansions of the solutions have the form

$$\sigma = \sigma_0 p^2 + \sum_{n=1}^{\infty} \sigma_n p^{2-n/2}, \quad \tau = \tau_0 p^2 + \sum_{n=1}^{\infty} \tau_n p^{2-n/2} \tag{2.24}$$

For simplicity, we will consider only the single point $x=5/2$, $y=5/3$ on curve (2.23). For $s=s_0=-1/2$, we seek σ_1 and τ_1 in the expansion (2.24) in the form (Ref. 6, Lemma 2.2 and Example 2.3)

$$(\sigma_1, \tau_1) = B_0 + B_1 \ln p; \quad B_0 = (b_{01}, b_{02}), \quad B_1 = (b_{11}, b_{12})$$

Here the constants b_{01} and b_{11} are arbitrary, $b_{02}=-(25/9)b_{11}$, and $b_{12}=0$ (Ref. 6, Section 3.5). On curve (2.23), away from the point considered the constants b_{01} and b_{11} remain arbitrary, and b_{02} and b_{12} are expressed in terms of these constants and x. Thus, here, too, there is a two-parameter logarithm.

Case 3 *(Ref. 12, Section 8.2).* Here $x\neq 1$, $x\neq y$, and the expansion of the solutions has the form (2.24), where

$$\sigma_0 = (1-x)/y, \quad \tau_0 = x-y, \quad \sigma_1 \text{ is an arbitary Constant } \tau_1 = -y^2\sigma_1 \tag{2.25}$$

For $n=2$, we seek $(\sigma_2, \tau_2)=B_0+B_1 \ln p+B_2(\ln p)^2$, where $B_i=(b_{i1}, b_{i2})$, $i=0, 1, 2$. We have (Ref. 6, Section 3.6)

$$\begin{aligned} b_{21} &= \frac{xy^2\sigma_1^2}{4(x-y)(x-1)}, \quad b_{22} = -\frac{xy^3\sigma_1^2}{4(x-y)(x-1)}, \quad b_{12} = -yb_{11} - \frac{(y-1)y^3\sigma_1^2}{2(x-y)(x-1)} \\ b_{02} &= -yb_{01} + \frac{y(y-1)}{x}b_{11} + \frac{(y-1)z}{x} - \frac{(y-1)y^3\sigma_1^2}{2x(x-y)} \end{aligned} \tag{2.26}$$

Here b_{01} and b_{11} are arbitrary constants. When $s=-3$, i.e., $n=3$, the compatibility condition holds (because the critical eigenvalue $s_3=-3$ is not dangerous). Therefore, the coefficients σ_3 and τ_3 are second-degree polynomials in $\ln p$ and have one arbitrary parameter. According to Theorem 2.3 in Ref. 6, in the expansions (2.24) the degree of the polynomials σ_n and τ_n in $\ln p$ does not exceed n.

The presence of a two-parameter logarithm prevents the existence of an additional analytic first integral in system (2.8). In Case 3, it is absent in the domain of definition of the family F_{22}, i.e., when $x\neq y$ and $x\neq 1$. In Case 1, it is absent on curve (2.20). This curve intersects the straight line $x=y$ at the point $x=y=22$. Therefore, the additional analytic first integral is also absent at this point.

2.6. Non-power-law asymptotic forms

The six families G_1-G_6 of non-power-law asymptotic forms were found.[5,14,18] The family G_1 was determined on the curve $x=2y(y-1)/(y+2)$ (Fig. 2, $\alpha=-1$). For this family,

$$\begin{aligned} \sigma &= \frac{c}{p}\left[\ln p - \left(\beta_1 + \frac{6}{5}(1+\ln\ln p)\right) + O\left(\frac{1}{\ln p}\right)\right] \\ \tau &= -\frac{2yp^2}{y+2}\left[1 + \frac{2}{\ln p} + \frac{1}{(\ln p)^2}\left(3 + 2\beta_1 + \frac{12}{5}\ln\ln p\right) + O\left(\frac{1}{(\ln p)^3}\right)\right] \end{aligned} \tag{2.27}$$

where c and β_1 are arbitrary constants.

The family G_3 was determined on the curve $x=16y(y-1)/(8y-9)$ (Figs. 2 and 4, the family H_9, $\alpha=4$) and has the form

$$\begin{aligned} \sigma &= \frac{cp^4}{(\ln p)^4}\left[1 - \frac{1}{\ln p}\left(\beta_1 + \frac{13}{5}(1+\ln\ln p)\right) + O\left(\frac{1}{(\ln p)^2}\right)\right] \\ \tau &= -\frac{yp^2}{8y-9}\left[1 - \frac{2}{\ln p} + \frac{1}{(\ln p)^2}\left(3 - \frac{\beta_1}{2} - \frac{13}{10}\ln\ln p\right) + O\left(\frac{1}{(\ln p)^3}\right)\right] \end{aligned} \tag{2.28}$$

where c and β_1 are arbitrary constants.

The family G_5 was determined for $x=2$, $y\neq 2$ (see Fig. 3, $x=2$) and has the form

$$\begin{aligned}\sigma &= p^2\left[-\frac{1}{\kappa_0}+\frac{\kappa_1}{\kappa_0^3\ln p}+\frac{\kappa_2-y\kappa_0\beta_2+\kappa_3\ln\ln p}{\kappa_0^5(\ln p)^2}+O\left(\frac{1}{(\ln p)^3}\right)\right]\\ \tau &= p^2\left[\eta_0\ln p+\beta_2+\eta_1\ln\ln p+O\left(\frac{1}{\ln p}\right)\right]\end{aligned} \tag{2.29}$$

where

$$\kappa_0 = 2y-2,\quad \kappa_1 = (y-2)(3y-2),\quad \kappa_2 = (y-2)(3y-2)(6y^2-10y+5)$$

$$\kappa_3 = -y(y-2)(3y-2)^2(3y-4),\quad \eta_0 = 2(y-1)(y-2)/y$$

$$\eta_1 = (y-2)(3y-2)(3y-4)/(2y-2)$$

and β_2 is an arbitrary constant.

The families G_2, G_4 and G_6 are symmetric to the families G_1, G_3 and G_5, respectively, according to (2.14). They were determined on the lines $x=2(1-y)/(2y+1)$, $x=16(y-1)/(9y-8)$ and $x=2y$, $x\neq 1$, respectively.

2.7. Compound expansions

The non-power-law asymptotic form (2.27) corresponds to the two-parameter family (in c and β_1) of the compound expansions[20]

$$\sigma = p^{-1}\left(\sigma_0+\sum_{s=1}^{\infty}\sigma_s p^s\right),\quad \tau = p^2\left(\tau_0+\sum_{s=1}^{\infty}\tau_s p^s\right) \tag{2.30}$$

where σ_s and τ_s are uniquely defined series in decreasing integer degrees of $\ln p$, whose coefficients are polynomials in the double logarithm $\ln\ln p$, and $p^{-1}\sigma_0$ and $p^2\tau_0$ have the form indicated in (2.27) for σ and τ.

Similarly, the non-power-law asymptotic form (2.28) corresponds to the two-parameter family (in c and β_1) of compound expansions

$$\sigma = p^4\left(\sigma_0+\sum_{s=1}^{\infty}\sigma_s p^{-2s}\right),\quad \tau = p^2\left(\tau_0+\sum_{s=1}^{\infty}\tau_s p^{-2s}\right) \tag{2.31}$$

where σ_s and τ_s are the same as in the preceding case, and $p^4\sigma_0$ and $p^2\tau_0$ are indicated in (2.28) as σ and τ.

For the non-power-law asymptotic form (2.29), the existence of a compound expansion like (2.30) and (2.31) cannot yet be claimed.[20] Here the expansion has a more complicated structure than expansions (2.30) and (2.31).

2.8. Back to the system (1.1)[13,14,18]

According to formulae (2.9), solutions of system (1.1) can be obtained from the solutions of system (2.8). The expansions (2.18) and (2.19) of the solutions of system (2.8) correspond to the expansions of the solutions of system (1.1)

$$\begin{aligned}&p = t^{n_1}\sum p_s t^s,\quad q = t^{n_2}\sum q_s t^s,\quad r = t^{n_3}\sum r_s t^s,\quad p_0, q_0, r_0\neq 0\\ &\gamma_i = t^{m_i}\sum g_{is}t^s,\quad g_{i0}\neq 0,\quad i = 1,2,3\end{aligned} \tag{2.32}$$

where Re $s\geq 0$ if $t\to 0$, and Re $s\leq 0$ if $t\to\infty$. If $\alpha+\beta\neq 2$, the form of the expansion is maintained, i.e., a power expansion remains a power expansion, a power logarithmic expansion remains a power logarithmic expansion, and a compound expansion remains a compound expansion. We will use H_k' to denote the family of expansions of the

Table 3

k	N	M	Λ
1	$2, 0, 1$	$0, 0, 1$	$-2, 0, 0, 1, 2$
3	$1, 0, 0$	$0, 0, 0$	$1, 0, 0, 0, 2$
4	$3, 1, 1$	$0, 0, 0$	$-3, 0, -1, 0, 2$
5	$2, -1, 2$	$-2, 1, -2$	$\frac{5+\varphi_1}{2}, \frac{\varphi_1-5}{2}, 0, 4, 2$
7	$-\frac{2}{\alpha}, -1, -\frac{2}{\alpha}$	$-2, -\frac{2}{\alpha}-1, -2$	$\frac{4}{\alpha}-1, 0, 2+\frac{2}{\alpha}, 4, 2$
9	$-\frac{2}{\alpha}, -1, -\frac{2}{\alpha}$	$-2, -\frac{2}{\alpha}-1, -2$	$\frac{4}{\alpha}-1, 0, 2+\frac{2}{\alpha}, 4, 2$
11	$-3, -3, -1$	$-2, -2, 0$	$\frac{1+\varphi_3}{2}, \frac{1-\varphi_3}{2}, 1, 0, 2$
13	$-\frac{2}{\beta}, \frac{2}{\beta}, -1$	$-2, -2, 1$	$0, \frac{2}{\beta}-3, 0, 2-\frac{2}{\beta}, 2$
15	$-\frac{2}{\beta}, \frac{2}{\beta}, -1$	$-2, -2, \frac{2}{\beta}-2$	$0, 0, 3-\frac{2}{\beta}, 5-\frac{4}{\beta}, 2$
17	$-\frac{2}{\beta}, \frac{2}{\beta}, -1$	$-2, -2, 3-\frac{2}{\beta}$	$0, \frac{4}{\beta}-5, \frac{2}{\beta}-2, 0, 2$
19	$-1, -1, -1$	$-2, -2, -2$	$\frac{1+\varphi_5}{2}, \frac{1-\varphi_5}{2}, 3, 4, 2$
20	$-1, -1, -1$	$-2, -2, -1$	$1, 0, 3, 4, 2$

solutions of system (1.1) obtained from the family H_k of solutions of system (2.8). Each of the families H'_k has the six eigenvalues $\lambda_1, \ldots \lambda_5$, and $\lambda_6 = -1$. They are also called the Kovalevskaya exponents. Table 3 presents the values of $N = (n_1, n_2, n_3)$, $M = (m_1, m_2, m_3)$ and $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ for the families H'_k. The points $x = y = 2$ and $x = 1$, $y = 2$ are the only points where all the expansions (2.32) do not contain logarithms and have integer exponents, and there is an additional Kovalevskaya first integral (1.7).

The families H_{23} and H_{24} have the sum $\alpha + \beta = 2$ and the product $\sigma\tau = \sigma_0\tau_0 p^2 + \ldots$. Therefore, according to the last formula (2.9), they have $t = \mathrm{const}\cdot\ln p + \ldots$, i.e., t is not a power of p, and the corresponding expansions of solutions of system (1.1) will not be power expansions.

The family H'_{11} was calculated by Appelrott.[21] The families H'_{19} and H'_{20} were calculated by Kovalevskaya,[22] who incorrectly indicated the eigenvalue $\lambda_3 = 0$ for the family H'_{19}. The exponents N, M and the eigenvalues Λ were calculated for several cases by Gashenenko.[23]

3. Simple exact solutions of the Kowalewski equations

3.1. Introduction

For the Kowalewski Eq. (2.8) there are nine known families of particular solutions (Steklov,[24] Goryachev,[25] Chaplygin,[26] Kowalewski,[3] Appelrott,[27] Gorr,[28,29] and Dokshevich and Konosevich–Pozdnyakovich).[30] In the non-integrable cases, all the known particular solutions are finite sums of rational powers of variables of three types: a) p, b) $p + \mathrm{const}$, and c) $p^2 + \mathrm{const}$.

It is now possible to find all such solutions. According to the discussion in Section 2, the Kowalewski Eq. (2.8) have 24 families of logarithmic power expansions of the solutions. In 10 of the families, $p \to 0$ (we will call them *tails*), and in 14 families $p \to \infty$ (we will call them *heads*). The finite expansions of the solutions of type a are found by ascertaining which tail–head pairs are compatible, i.e., give a finite expansion, and which do not. All the particular solutions of type a, including all seven known solutions[3,24–29] and five more new solutions,[31–33] were obtained in this way. All the new solutions are complex. We will next prove that there are no other solutions in the form of finite sums of rational powers of p with complex coefficients.

3.2. Statement of the problem

The same numbering of the families of expansions of the solutions of the Kowalewski equations that was used in Section 2 is used here. In fact, only power expansions with rational power exponents are used because the expansions with complex or irrational real power exponents, as well as with logarithms cannot terminate. The complex solutions of the Kowalewski equations that correspond to complex solutions of the Euler–Poisson equations are considered. System (2.8) has the two first integrals (2.10). In (2.8) and (2.10) the coefficients a_i, b_i, c_i and d_i are to the rational functions (2.12) of the parameters (2.11), where h and l are values of the energy and momentum integrals (1.3) for the Euler–Poisson equations (1.1). Here x and y are real and satisfy inequalities (2.13), which define the set **D**; $z, \lambda \in \mathbb{C}$; $\xi \in \mathbb{R}$, $\xi \neq 0$. Systems (2.8) and (2.10) have the symmetry transformation (2.14).

Problem. It is required to find all the solutions $\sigma(p)$ and $\tau(p)$ of the system (2.8) that are finite sums of rational powers of p:

$$\sigma(p) = \sum_{k=0}^{m} \sigma_k p^{\alpha_k}, \quad \tau(p) = \sum_{l=0}^{n} \tau_l p^{\beta_l} \tag{3.1}$$

where α_k and β_l are rational numbers, the constants $\sigma_k, \tau_l \in \mathbb{C}$, and $\sigma_0, \sigma_m, \tau_0, \tau_n \neq 0$.

The solutions of the Kowalewski Eq. (2.8) that correspond to real solutions of the Euler–Poisson equations, i.e., the solutions for which $\lambda, z \in \mathbb{R}$ and $(y-1)\sigma$, $(y-1)\tau \geq 0$, are considered as real solutions. The finite solution (3.1) is considered known if it was published somewhere or if it can be obtained from a published solution by applying the symmetry transformation (2.14) or by taking into account another root of the algebraic equation that specifies the value of the specific parameter. A solution that belongs to a boundary of a (generating) family of solutions, i.e., that lies within it, is not considered as an independent solution.

3.3. Method

The problem was solved by using the list of all the 23 families $H_1 - H_{21}, H_{23}, H_{24}$ of power expansions of the solutions of system (2.8)

$$\sigma = \sigma_0 p^{\alpha} + \sum \sigma_{\alpha+s} p^{\alpha+s}, \quad \tau = \tau_0 p^{\beta} + \sum \tau_{\beta+s} p^{\beta+s}, \quad s \in \mathbf{K} \tag{3.2}$$

where

$$\alpha, \beta, s \in \mathbb{R}; \quad \sigma_0, \tau_0, \sigma_{\alpha+s}, \tau_{\beta+s} = \text{const} \in \mathbb{C}; \quad \sigma_0, \tau_0 \neq 0$$

The families $H_1 - H_{21}$ were found in Refs. 10–18, and H_{23} and H_{24} were found in Ref. 19, where it was also shown that there are no other expansions. This list together with the family H_j also contains the family $\bar{H}_j$, which is symmetric to it according to (2.14). Usually $\bar{H}_j \neq H$, and only $H_3 = \bar{H}_3$, $H_4 = \bar{H}_4$ and $H_{19} = \bar{H}_{19}$.

In the 10 families $H_1 - H_8, H_{23}, H_{24}$ the variable $p \to 0$; as above, we will call them tails. In the 13 families $H_9 - H_{21}$ the variable $p \to \infty$; we will call them heads. Each finite expansion (3.1) has one tail and one head. Therefore, for each pair of families consisting of a tail H_i and a head H_j, we must study the intersection

$$\mathcal{H}_i \cap \mathcal{H}_j; \quad i \in 1, 2, \ldots, 8, 23, 24; \quad j \in 9, 10, \ldots, 21 \tag{3.3}$$

If it is not empty, it gives a finite expansion (3.1). If it is empty, an expansion (3.1) with such a tail and a head does not exist. This approach enables us to find all the finite expansions (3.1).

The intersections (3.3) are analysed in the following way. For each family H_m of the expansions (3.2), the following are known:

$$\alpha^{(m)} \overset{\text{def}}{=} \alpha, \quad \beta^{(m)} \overset{\text{def}}{=} \beta;$$

the set $\mathbf{K}^{(m)} \stackrel{\text{def}}{=} \mathbf{K}$ of values of s, i.e., the sets $\mathbf{K}_\sigma^{(m)}$ and $\mathbf{K}_\tau^{(m)}$ of the power exponents $\alpha + s$ and $\beta + s$;
the set $\mathbf{M}^{(m)}$ of admissible values of the parameters x, y, z, λ and ξ;
the arbitrary coefficients among $\sigma_{\alpha+s}$ and $\tau_{\beta+s}$.

For each family H_m, any finite number of the coefficients $\sigma_{\alpha+s}$ and $\tau_{\beta+s}$ in expansion (3.2) for it can be computed as rational functions of the parameters.

The conditions

$$\alpha^{(i)} \le \alpha^{(j)}, \quad \beta^{(i)} \le \beta^{(j)}$$

$$\mathbf{K}_\sigma^{(i)} \cap \mathbf{K}_\sigma^{(j)} \ne \varnothing, \quad \mathbf{K}_\tau^{(i)} \cap \mathbf{K}_\tau^{(j)} \ne \varnothing, \quad \mathbf{M}^{(i)} \cap \mathbf{M}^{(j)} \ne \varnothing$$

are necessary for a non-empty intersection (3.3) to exist.

In the next step, the possibility of the equality $\sigma_{\alpha^{(i)}} = \sigma_{\alpha^{(j)}+s^{(j)}}$ is ascertained for each pair $\alpha^{(i)} + s^{(i)} = \alpha^{(j)} + s^{(j)}$. The possibility of the analogous equality for $\tau_{\beta+s}$ is also ascertained.

3.4. Results

In this way, 30 families of finite solutions of the type (3.1) were obtained. Altogether, there are 16 basic families of solutions, which are denoted by $R_1 - R_{16}$, and 14 more solutions that are symmetric to them according to (2.14) because R_7 and R_{13} are symmetric to themselves: $R_7 = \bar{R}_7$, $R_{13} = \bar{R}_{13}$.

The new families are

$$\mathcal{R}_1: x = y = 2, \quad z = 0, \quad \lambda \ne 0$$

$$\sigma = \frac{\varepsilon\lambda}{8} p^{-1} + \frac{\xi}{2\lambda} p - \frac{1}{2} p^2, \quad \tau = -2p^2$$

$$\mathcal{R}_2: y = 1 + x/2, \quad z = \lambda = 0$$

$$\sigma = -\frac{2\xi^2}{(x+1)(x-1)^2} p^{-2} + \frac{1-x}{2} p^2, \quad \tau = -\frac{1}{2} p^2$$

$$\mathcal{R}_3: x = y = 2, \quad z = \lambda = 0$$

$$\sigma = -\frac{2\xi^2}{3} p^{-2} - \frac{1}{2} p^2, \quad \tau = -\frac{1}{2} p^2$$

$$\mathcal{R}_5: x = y, \quad \lambda = z = 0$$

$$\sigma = \sigma_0 p^{2/3} + \sigma_2 p^2, \quad \tau = \tau_0 p^{2/3}$$

$$\tau_0^3 = \frac{81 y \xi^2}{y+2}, \quad \frac{\sigma_0}{\tau_0} = -\frac{y+2}{3y^2}, \quad \sigma_2 = \frac{1-y}{y}$$

$$\mathcal{R}_{15}: x = \frac{8}{5}, \quad y = \frac{9}{5}, \quad z = \lambda = 0$$

$$\sigma = \frac{125\xi^2}{288} p^{-2} - \frac{1}{18} p^2, \quad \tau = -\frac{1}{2} p^2 - \frac{88}{625\xi^2} p^6$$

$$\mathcal{R}_{16}: x = \frac{14}{9}, \quad y = \frac{16}{9}, \quad z = -\frac{11}{36\tau_1}, \quad \lambda = 0$$

$$\sigma = -\frac{11}{1152\tau_1^2} p^{-2} - \frac{11}{144\tau_1} - \frac{1}{8} p^2, \quad \tau = -\frac{1}{2} p^2 - \tau_1 p^4$$

$$\tau_1^2 = -\frac{24167}{6^8 \xi^2}$$

Here $R_3 \subset R_2$. All these solutions are complex. The solutions R_1 and R_3 are assigned to the integrable Kovalevskaya case (1.7), and R_5 is assigned to the integrable Chaplygin case when $y=4$.

Gorr[28] found a complex solution, which can be written in the detailed form

$$\mathcal{R}_4\colon x = \frac{1268 - 44\sqrt{409}}{375}, \quad y = \frac{241 - 3\sqrt{409}}{250}, \quad z = 0, \quad \lambda = 0$$

$$\sigma = \sigma_0 p^{10/3} + \sigma_2 p^2 + \sigma_4 p^{2/3}, \quad \tau = \tau_0 p^2 + \tau_2 p^{2/3}$$

where σ_0 satisfies the equation

$$\sigma_0^3 \xi^2 = \frac{997140.82763947063\sqrt{409} - 2016592.23367734611}{13056} < 0$$

and the other coefficients are

$$\sigma_2 = \frac{16459 - 847\sqrt{409}}{4352}, \quad \sigma_4 = \frac{1211697549 - 59858217\sqrt{409}}{189399040\sigma_0}$$

$$\tau_0 = \frac{7\sqrt{409} - 109}{100}, \quad \tau_2 = \frac{2521\sqrt{409} - 51037}{16000\sigma_0}$$

However, the solution R_5, which is written above, was missing.

All the other known families of solutions have real parts. The family R_8 belongs to the closure of the family R_{11}, and $R_9, R_{10} \subset R_{14}$. The solutions R_9, R_{10}, R_{14} are assigned to the integrable Kovalevskaya case (1.7), and R_6 is assigned to the integrable Chaplygin case.

If the finite expansion (3.1) is written in the form

$$\sigma = \sigma_0 p^{\alpha_1} + \ldots + \sigma_n p^{\alpha_2}, \quad \tau = \tau_0 p^{\beta_1} + \ldots + \tau_m p^{\beta_2}$$

where $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, the results presented above can be represented as in Table 4, which also gives the data of other authors.[3,24–29,34] Its columns correspond to the basic tails (without the symmetric tails), and its rows correspond to all the heads. The intersection of the i-th column and the j-th row has the index k of the family $R_k = \mathcal{H}_i \cap H_j$. References are given for the known solutions in square brackets. An empty cell in the table corresponds to the empty intersection $H_i \cap H_j$. A bar over the index k, i.e., $\bar{k}$, or over a reference denotes the symmetric solution $\bar{R}_k$ according to (2.14) or the solution that is symmetric to the solution presented in the reference. Since the intersections of the heads H_{13}, H_{14}, H_{17} and H_{18} with all the tails and the intersections of the tail H_{23} with all the heads are empty, Table 4 does not contain rows and a column that correspond to these families.

Table 4

Heads		Tails				
		H_1	H_3	H_4	H_5	H_7
		$\alpha_1=0$	$\alpha_1=0$	$\alpha_1=2/3$	$\alpha_1=-1$	$\alpha_1<0$
		$\beta_1=1$	$\beta_1=0$	$\beta_1=2/3$	$\beta_1=2$	$\beta_1=2$
H_9	$\alpha_2>2,\ \beta_2=2$		11 [3], 12 [25]	4 [28]		
H_{10}	$\alpha_2=2,\ \beta_2>2$	8 [3]	$\overline{11}$ $\overline{[3]}$, $\overline{12}$ $\overline{[25]}$	$\bar{4}$ $\overline{[28]}$		15, 16
H_{11}	$\alpha_2=2,\ \beta_2=2/3$			5		
H_{12}	$\alpha_2=2/3,\ \beta_2=2$			5		
H_{15}	$\alpha_2=2,\ \beta_2\in(1,2)$			6 [29]		
H_{16}	$\alpha_2\in(1,2),\ \beta_2=2$			$\bar{6}$ $\overline{[29]}$		
H_{19}	$\alpha_2=\beta_2=2$		13 [24]	7 [26]		2
H_{20}	$\alpha_2=\beta_2=2$	10 [27]	14 [27]		1	3
H_{21}	$\alpha_2=\beta_2=2$	9 [34]	$\overline{14}$ $\overline{[27]}$			

We will now formulate the main result.

Theorem 3.1. *The system of* Eq. (2.8) *has only the exact solutions of the form* (3.1), *as indicated in* Table 4.

A detailed proof can be found in the preprint Ref. 31.

4. The local integrability of the Euler–Poisson equations

4.1. Local integrability

Consider an autonomous system of ordinary differential equations (the prime denotes a derivative with respect to t)

$$x_j' = \varphi_j(X), \quad j = 1, \dots, n \tag{4.1}$$

where $X = (x_1, \dots, x_n)$ and the φ_j are polynomials. Such a system is defined in the n-dimensional complex phase space $\mathbb{C}^n$. We will say that system (4.1) is *locally integrable* in the domain $D \subset \mathbb{C}^n$ if in this domain it has the necessary number of independent first integrals of the form $a_i(X)$, where the functions $a_i(X)$ are analytic in the domain D. Of course, according to Cauchy's theorem, in some vicinity of the non-stationary point X^0, where there is at least one function for which $\varphi_j(X^0) \neq 0$, system (4.1) has $n-1$ analytic first integrals, i.e., is integrable. Therefore, the question of integrability is interesting for domains that contain such singularities as a stationary solution (a fixed point) or a periodic solution.

Thus, the local integrability of system (4.1) can be studied near its singular points. Heretofore, only its global integrability over the entire phase space was studied. The first attempts of this kind were undertaken in Refs. 35–37 and were continued in Refs. 38–40.

Below all vectors are denoted by capital letters and are written down as row matrices, and an asterisk denotes the transposition.

4.2. Normal forms

The local integrability (or non-integrability) of a system of ordinary differential equations near a singularity can be established most simply using its normal form (Ref. 41, Chapter 3; Ref. 4, Chapter 5, Section 6). This is true for singularities such as a stationary point, a periodic solution and an invariant torus. Here we refer only to the normal form of the system in the vicinity of its stationary point $X = 0$, where the system (4.1) has the form

$$X'^* = AX^* + \Phi^*(X) \tag{4.2}$$

Here A is a constant square matrix and the vector polynomial $\Phi(X) = (\varphi_1(X), \dots, \varphi_n(X))$ does not contain constant or linear terms. Suppose the linear substitution

$$X^* = BY^* \tag{4.3}$$

brings the matrix A into the Jordan form $J = B^{-1}AB$ and brings the entire system (4.2) into the form

$$Y'^* = JY^* + \tilde{\Phi}^*(Y) \tag{4.4}$$

Suppose the formal change of coordinates

$$Y = Z + \Xi(Z) \tag{4.5}$$

where $\Xi = (\xi_1, \dots \xi_n)$ and $\xi_j(Z)$ are formal power series without constant and linear terms, transforms the system (4.4) into the system

$$Z'^* = JZ^* + \Psi^*(Z) \tag{4.6}$$

where $\Psi(Z)$ is a vector power series without constant and linear terms. We will write it in the form

$$z_j' = z_j g_j(Z) \stackrel{\text{def}}{=} z_j \sum g_{jQ} Z^Q \quad \text{for} \quad Q \in \mathbb{N}_j, \quad j = 1, \dots, n \tag{4.7}$$

where

$$Q = (q_1, \ldots, q_n), \quad Z^Q = z_1^{q_1} \ldots z_n^{q_n}, \quad \mathbb{N}_j = \{Q : Q \in \mathbb{Z}^n, Q + E_j \geq 0\}$$

and E_j is the j-th unit vector. We set $\mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_n$. Since J is a Jordan matrix, its diagonal $\Lambda = (\lambda_1 \ldots, \lambda_n)$ consists of the eigenvalues λ_j of the matrix A.

System (4.6), (4.7) is called a *resonant normal form* if a) the matrix J is a Jordan matrix and b) (4.7) contains only the *resonant terms* $g_{jQ}Z^Q$ for which the scalar product

$$\langle Q, \Lambda \rangle \stackrel{\text{def}}{=} q_1\lambda_1 + \ldots + q_n\lambda_n = 0 \tag{4.8}$$

Theorem 4.1 ([42]). *A formal change (4.5) exists that brings the system (4.4) into the normal form (4.6), (4.7).*

The properties of the normal form and the normalizing transformation are described in Refs. 41,4 and in Ref. 43. Let k be the number of linearly independent solutions $Q = \mathbb{N}$ of Eq. (4.8), which is called *multiplicity* of the *resonance*. Integration of the normal form (4.7) can be reduced to solving a system of order k for the k *resonant variables*. Note the following

Property 1 (Ref. 4, Chapter 3, Section 1). *If system (4.4) has the linear automorphism $\tilde{t} = \delta t$, $Y^* = LY^*$, its normal form (4.6) has the analogous automorphism $\tilde{t} = \delta t$, $\tilde{Z}^* = LZ^*$.*

Bringing a system into its normal form involves the introduction of coordinates in which the system has the simplest form and its solutions are as straightened as possible. However, the normalizing transformation does not always converge and give an analytic change of the local coordinates. It often diverges and can be used only for an approximate description (with accuracy of any order) of the solutions near a stationary point. However, even in the case of divergence, it enables us to find families of periodic solutions and families of conditionally periodic solutions adjacent to the stationary point (Ref. 41, Chapter 3, Section 3).

The conditions imposed on the normal form (4.7) which ensure convergence of the normalizing transformation were pointed out in Ref. 43. We will formulate them for the cases encountered here.

Condition A. Two power series $\alpha(Z)$ and $\beta(Z)$ exist such that

$$g_j(Z) = \lambda_j\alpha(Z) + \lambda_j\beta(Z), \quad j = 1, 2, \ldots, n$$

in the normal form (4.7).

We set

$$\omega_k = \min|\langle Q, \Lambda \rangle| \text{ по } Q : Q \in \mathbb{N}, \quad \langle Q, \Lambda \rangle \neq 0, \quad \sum q_j < 2^k$$

Condition ω. The series

$$\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k > -\infty$$

converges.

Theorem 4.2 ([43]). *If in the analytic system (4.4) the vector Λ satisfies condition ω and the normal form (4.6), (4.7) satisfies condition A, the normalizing transformation (4.5) is analytic for sufficiently small $|z_j|$.*

4.3. Stationary points

Consider system (1.1) with

$$A = B, \quad Mgx_0/B = 1, \quad y_0 = z_0 = 0, \quad C/B = c \tag{4.9}$$

Then system (1.1) takes the form

$$\begin{aligned} &p' = (1-c)qr, \quad q' = (c-1)pr - \gamma_3, \quad r' = \gamma_2/c \\ &\gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' = p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2 \end{aligned} \tag{4.10}$$

System (4.10) has one parameter $c \in (0, 2]$. This half-open interval corresponds to the values of c in the mechanical problem. System (1.1) is integrable in quadratures if it contains an additional first integral.[2] System (4.10) has an additional first integral in only two cases: the case in which $c = 1$ (the Lagrange–Poisson case) and the case in which $c = 1/2$ (the Kovalevskaya case).

Theorem 4.3. *System (4.10) has two pairs of two-parameter families of stationary points*

$$\mathbf{S}_\sigma: p = p_0 \in \mathbb{C}, \quad q = 0, \quad r = 0, \quad \gamma_1 = \sigma, \quad \gamma_2 = 0, \quad \gamma_3 = 0, \quad \sigma = \pm 1 \tag{4.11}$$

$$\begin{gathered} \mathbf{T}_\tau: p = p_t \in \mathbb{C}, \quad q = 0, \quad r_t = \tau\sqrt{1-(c-1)^2 p^4}/((c-1)p) \\ \gamma_{1t} = (c-1)p_t^2, \quad \gamma_2 = 0, \quad \gamma_{3t} = \tau\sqrt{1-(c-1)^2 p^4}, \quad \tau = \pm 1 \end{gathered} \tag{4.12}$$

and all its stationary points belong to these families.

This trivial result can be derived both directly from system (4.10) and from Staude's classical results.

The families $\mathbf{S}_\sigma$ exist for any $c \in \mathbb{C}$. The families $\mathbf{T}_\tau$ exist for $c \neq 1$ and $p_t \neq 0$. The intersections of the families are as follows:

$$\mathbf{S}_+ \cap \mathbf{S}_- = \varnothing$$

$$\begin{gathered} \mathbf{T}_+ \cap \mathbf{T}_- = \mathbf{S}_\sigma \cap \mathbf{T}_\tau : p_0 = p_t, \quad q = 0, \quad r = 0 \\ \gamma_1 = (c-1)p_t^2 = \sigma = \pm 1, \quad \gamma_2 = 0, \quad \gamma_3 = 0 \end{gathered}$$

The families $\mathbf{S}_\sigma$ are real if $p_0 \in \mathbb{R}$. The families $\mathbf{T}_\tau$ are real if $p_t \in \mathbb{R}$ and $(c-1)^2 p_t^4 \leq 1$.

4.4. The families S_σ

In the vicinity of each stationary point (4.11) we introduce the local coordinates

$$P = p - p_0, \quad q, r, \Gamma = \gamma_1 - \sigma, \gamma_2, \gamma_3 \tag{4.13}$$

System (4.10), written in the local coordinates (4.13), takes the form

$$\begin{aligned} P' &= (1-c)qr \\ q' &= (c-1)p_0 r - \gamma_3 + (c-1)Pr \\ r' &= \gamma_2/c \\ \Gamma' &= r\gamma_2 - q\gamma_3 \\ \gamma_2' &= -\sigma r + p_0\gamma_3 + P\gamma_3 - r\Gamma \\ \gamma_3' &= \sigma q - p_0\gamma_2 + q\Gamma - P\gamma_3 \end{aligned} \tag{4.14}$$

If the coordinates (4.13) are treated as $X = (x_1, \ldots, x_6)$:

$$x_1 = P, \quad x_2 = q, \quad x_3 = r, \quad x_4 = \Gamma, \quad x_5 = \gamma_2, \quad x_6 = \gamma_3 \tag{4.15}$$

then system (4.14) takes the form of system (4.2) with $n = 6$ and the matrix A:

$$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (c-1)p_0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma & 0 & 0 & p_0 \\ 0 & \sigma & 0 & 0 & -p_0 & 0 \end{Vmatrix} \tag{4.16}$$

Its characteristic equation

$$\lambda^6 + a\lambda^4 + b\lambda^2 = 0 \tag{4.17}$$

where

$$a = (\sigma p_0^2 + 1/c + 1)\sigma, \quad b = 1/c + \sigma p_0^2(1/c - 1) \tag{4.18}$$

has two zero roots and paired roots:

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -\lambda_4, \quad \lambda_5 = -\lambda_6 \tag{4.19}$$

where

$$2\lambda_3^2 = -a + \sqrt{a^2 - 4b}, \quad 2\lambda_5^2 = -a - \sqrt{a^2 - 4b}$$

We will examine the dependence of λ_3, λ_4, λ_5 and λ_6 on the two real parameters

$$x \overset{\text{def}}{=} 1/c \geq 1/2, \quad y \overset{\text{def}}{=} \sigma p_0^2 \in \mathbb{R}$$

According to equalities (4.18),

$$\sigma a = x + 1 + y, \quad b = x + y(x - 1) \tag{4.20}$$

When $a^2 - 4b \geq 0$, both λ_3^2 and λ_5^2 are real. After some elementary transformations, the equation $a^2 = 4b$ takes the form

$$(1 - x)^2 + 2y(3 - x) + y^2 \equiv (1 - x + y)^2 + 4y = 0$$

In the x, y plane it defines a parabola, which we denote by C_1 (see Fig. 5). To the left of it $a^2 - 4b < 0$. We denote this set by D_1. In it $\lambda_3^2 = \overline{\lambda_5^2}$, and $\operatorname{Im} \lambda_3^2 \neq 0$. Then $\lambda_3 = \overline{\lambda_5}$, and $\operatorname{Re} \lambda_3 \neq 0$, $\operatorname{Im} \lambda_3 \neq 0$. Therefore, $\operatorname{Im}(\lambda_3/\lambda_5) \neq 0$, i.e., the ratio λ_3/λ_5 is not real.

Now consider the part of the x, y plane where $a^2 - 4b > 0$. In this part, both λ_3^2 and λ_5^2 are real. The sign of their ratio is the same as the sign of b, i.e., when $b < 0$, the ratio λ_3/λ_5 is pure imaginary and non-zero. According to (4.20),

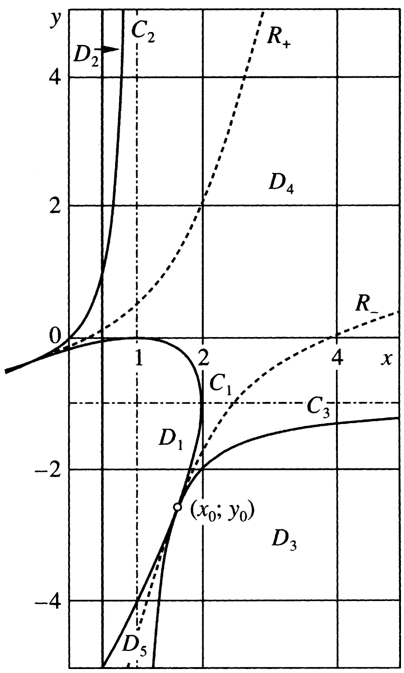

Fig. 5.

the equation $b = 0$ defines the hyperbola $x + y(x - 1) = 0$ with asymptotes $x = 1$ and $y = -1$; its upper branch C_2 and its lower branch C_3 are shown in Fig. 5. In the half-plane $x \geq 1/2$, C_2 does not meet the parabola C_1, but C_3 is tangent to C_1 at the point where $a = b = 0$, i.e., at the point with coordinates

$$x = x_0 \overset{\text{def}}{=} \frac{1+\sqrt{5}}{2} = 1.618\ldots, \quad y = y_0 \overset{\text{def}}{=} -\frac{3+\sqrt{5}}{2} = -x_0^2 = -2.618\ldots \tag{4.21}$$

The parts of the half-plane $x \geq 1/2$ that are located above the curve C_2 and below the curve C_3 are denoted by D_2 and D_3, respectively (Fig. 5). In them the ratio λ_3/λ_5 is pure imaginary. Outside the sets D_1, D_2, D_3 the half-plane $x \geq 1/2$ consists of the two parts D_4 and D_5, which are bounded by the curves C_1, C_2, C_3 with the set D_4 being located above the point (4.21) and D_5 below it (Fig. 5). In D_4 and D_5 the ratio λ_3/λ_5 is real. Note that the curves C_1, C_2, C_3 only bound the sets $D_1, \ldots, D_5$ and do not enter them. Thus, the following statement has been proved.

Lemma 4.1. *In the set D_1 the eigenvalues λ_3, λ_4, λ_5 and λ_6 are complex: $\lambda_4 = -\lambda_3$, $\lambda_5 = \bar{\lambda}_3$, $\lambda_6 = -\bar{\lambda}_3$. In D_2 and D_3 two of them are real, and two are pure imaginary. In D_4 and D_5 they are all either pure imaginary or real.*

These results trivially follow from the results of Rumyantsev and his students.[44–46]

Let $\lambda \neq 0$ be the root of Eq. (4.17). The corresponding eigenvector of matrix (4.16) is

$$(b_3/p_0)(0, \lambda(\sigma p_0^2 c + \lambda^2 c + \sigma), p_0, 0, \lambda c p_0, \lambda^2 c + \sigma) \tag{4.22}$$

where $b_3 \neq 0$ is any number. When the sign in front of λ is reversed, only the second and fifth components change: they change sign. Therefore, the following statement holds.

Lemma 4.2. *In the sets $D_1, \ldots, D_5$ the transformation* (4.3), *which converts matrix* (4.16) *into a diagonal matrix with the diagonal*

$$\Lambda = (0, 0, \lambda_3, -\lambda_3, \lambda_5, -\lambda_5) \tag{4.23}$$

has the form

$$\begin{aligned}
x_1 &= y_2 \\
x_2 &= t_1 y_3 - t_1 y_4 + \tau_1 y_5 - \tau_1 y_6 \\
x_3 &= t_2 y_3 + t_2 y_4 + \tau_2 y_5 + \tau_2 y_6 \\
x_4 &= y_1 \\
x_5 &= t_3 y_3 - t_3 y_4 + \tau_3 y_5 - \tau_3 y_6 \\
x_6 &= t_4 y_3 + t_4 y_4 + \tau_4 y_5 + \tau_4 y_6
\end{aligned} \tag{4.24}$$

In fact, the columns of the matrix B from (4.3) *are the eigenvectors of matrix A from* (4.2), *i.e.,* (4.16), *which have the form* (4.22).

System (4.14) has the automorphism

$$t, P, q, r, \Gamma, \gamma_2, \gamma_3 \to -t, P, -q, r, \Gamma, -\gamma_2, \gamma_3 \tag{4.25}$$

After the transformation (4.24), this automorphism in the coordinates (4.15) takes the form

$$t, y_1, y_2, y_3, y_4, y_5, y_6 \to -t, y_1, y_2, y_4, y_3, y_6, y_5 \tag{4.26}$$

The proof is simple in the reverse direction, that is, from (4.26) to (4.25).

Thus, the following statement has been proved.

Lemma 4.3. *In the sets $D_1, \ldots, D_5$ system* (4.14) *is transformed by the linear substitution* (4.24), *which diagonalises the matrix of its linear part, into system* (4.4), *which has the automorphism* (4.26).

4.5. The existence of four integrals

Theorem 4.4. *In the sets D_1, D_2 and D_3 the normalizing transformation (4.5) converges for sufficiently small $|z_j|$.*

Proof. According to Lemma 4.1, in these sets the ratio λ_3/λ_5 is not a real number; therefore, Eq. (4.8), i.e., $\langle Q, \Lambda\rangle = 0$, with the vector (4.23) has only the real solutions $Q = (q_1, \ldots, q_6)$, in which q_1 and q_2 have arbitrary values, $q_3 = q_4$, and $q_5 = q_6$. According to the definition of the set $\mathbb{N}$ in Subsection 4.2, if $Q \in \mathbb{N}$, all the components q_j are integers, $q_1 \geq -1$ and then $q_2 \geq 0$ or $q_2 \geq -1$ and then $q_1 \geq 0$ and $q_3, q_5 \geq 0$. Therefore, in the normal form (4.7) with $n = 6$, the series g_j depend only on

$$z_1, z_2, \rho_1 \overset{\text{def}}{=} z_3 z_4, \rho_2 \overset{\text{def}}{=} z_5 z_6 \tag{4.27}$$

which are resonant variables here. Therefore, the normal form (4.7) has the form

$$\dot{z}_j = z_j g_j(z_1, z_2, \rho_1, \rho_2), \quad j = 1, \ldots, 6 \tag{4.28}$$

According to Property 1 and Lemma 4.3, the normal form (4.28) has an automorphism similar to (4.26) with the y_j replaced by z_j. For the resonant variables (4.27), from the automorphism (4.26) we obtain the automorphism

$$t, z_1, z_2, \rho_1, \rho_2 \to -t, z_1, z_2, \rho_1, \rho_2$$

Therefore, in the normal form (4.28)

$$g_1 \equiv g_2 \equiv 0, \quad g_4 = -g_3, \quad g_6 = -g_5 \tag{4.29}$$

Now, let us ascertain in which cases condition A, i.e.,

$$g_j = \lambda_j \alpha + \bar{\lambda}_j \beta, \quad j = 1, \ldots, 6$$

holds in the normal form (4.28). According to (4.19) and (4.29), satisfaction of the two equalities

$$g_3 = \lambda_3 \alpha + \bar{\lambda}_3 \beta, \quad g_5 = \lambda_5 \alpha + \bar{\lambda}_5 \beta \tag{4.30}$$

is necessary and sufficient for this. These equalities comprise a system of linear equations for the series α and β. Using Lemma 4.1, we evaluate its determinant $\Delta = \lambda_3\bar{\lambda}_5 - \bar{\lambda}_3\lambda_5$ in the sets $D_1, \ldots, D_5$. In D_1 we have $\lambda_5 = \bar{\lambda}_3$; therefore, $\Delta = (\lambda_3 - \bar{\lambda}_3)(\lambda_3 + \bar{\lambda}_3) = 4i \operatorname{Re}\lambda_3 \operatorname{Im}\lambda_3 \neq 0$. In D_2 and D_3 we have $\lambda_3\lambda_5 = -\lambda_3\lambda_5$; therefore, $\Delta = 2\lambda_3\bar{\lambda}_5 \neq 0$. In D_4 and D_5 we have $\bar{\lambda}_3\lambda_5 = \lambda_3\bar{\lambda}_5$; therefore, $\Delta = 0$. Consequently, in D_1, D_2 and D_3 the system of Eq. (4.30) is solvable for α and β; therefore, the series α and β always exist, i.e., condition A holds. In D_4 and D_5 the system (4.30) cannot be solved, and condition A does not hold.

In the sets D_1, D_2 and D_3 condition ω is trivial, because if $\langle Q, \Lambda\rangle \neq 0$, for an integer Q

$$|\langle Q, \Lambda\rangle| \geq |\lambda_3||q_3 - q_4| + |\lambda_5||q_5 - q_6| \geq \min\{|\lambda_3|, |\lambda_5|\} > 0$$

According to Theorem 4.2, the normalizing transformation is analytic.

For the resonant variables (4.27), system (4.28), (4.29) gives the equations

$$\dot{z}_j = 0, \quad \dot{\rho}_j = 0, \quad j = 1, 2$$

Therefore, system (4.28), (4.29) has four independent first integrals

$$z_1 = \text{const}, \quad z_2 = \text{const}, \quad \rho_1 = \text{const}, \quad \rho_2 = \text{const} \tag{4.31}$$

Since the normalizing transformation is analytic and reversible, in sets D_1, D_2 and D_3 system (4.14) has four independent analytic local first integrals. Thus we obtain the following corollary. □

Corollary of Theorem 4.4. *In the sets D_1, D_2 and D_3 system (4.10) is locally integrable.*

Taking into account the signs of σ and p_0, we obtain 12 complex sets with local integrability.

If $p_0 = \pm\sqrt{y\sigma}$ is real, system (4.14) is real, and its normal form is also real in appropriate coordinates (Ref. 4, Chapter 5, Section 6, Property 4; see also Ref. 41, Chapter 3, Section 1). In particular, the four first integrals (4.31) are real. When $\sigma = +1$, $p_0 = \pm\sqrt{y}$ is real for $y \geq 0$, i.e., the set D_2 is real. When $\sigma = -1$, $p_0 = \pm\sqrt{-y}$ is real for $y \leq 0$, i.e., D_1 and D_3 are real. Altogether six real sets (taking into account the sign of p_0) with local integrability are obtained. For each value $x \geq 1/2$, i.e., $c \in (0, 2]$, there are real stationary points, near which the system is locally integrable.

A similar analysis for the families $\mathbf{T}_\tau$ (see (4.12)) showed that they have 32 complex sets with local integrability.[40] Among them four sets are real, and in all these sets $c < 3/4$.

4.6. Stationary points at infinity[36,47]

In Subsections 2.4 and 2.5, 24 families of power logarithmic expansions of solutions that have power-law asymptotic forms were found using power geometry for the Kowalewski Eq. (2.8). Among them, eight families of power expansions contain the largest possible number of arbitrary constants, i.e., the solutions of these families fill a domain in phase space in which there is a complete set of the first integrals that correspond to these arbitrary constants. However, these (local) first integrals can exist only in part of the phase space and not in the entire phase space. In these cases similar (local) first integrals exist in the Euler–Poisson system (1.1) and have the form $a(X)/b(X)$, where the functions $a(X)$ and $b(X)$ are analytic in the corresponding domain. The power transformation of the system of Euler–Poisson Eq. (1.1) was used to find these domains and analyse the structure of the phase space in them. It has been shown (Ref. 4, Chapter 3) that the power-law asymptotic form of a solution can be transformed into a stationary point using the power transformation

$$(\ln x_1, \ldots, \ln x_n)^* = \tilde{\alpha}(\ln y_1, \ldots, \ln y_n)^*$$

where $\tilde{\alpha}$ is a square matrix. This enables us to calculate the normal form in the vicinity of a power-law asymptotic form, as was done for 3 of the 24 families indicated. The normalization was performed by Edneral using a program that he wrote in the MATHEMATICA system.

In Subsection 2.3 we described the 24 families $F_1 - F_{24}$ of power-law asymptotic forms of the solutions of the Kowalewski Eq. (2.8) in the case $B \neq C$, $x_0 \neq 0$, $y_0 = z_0 = 0$. Among them there are 17 for $A = B$, i.e., for the case (4.9) with $c \neq 1$. Fifteen of these families F_k correspond to the families F_k of power-law asymptotic forms of the solutions of the Euler–Poisson Eq. (4.10) described in Section 2.8. To investigate the vicinities of the asymptotic forms of the family F_{10}, we perform the power transformation

$$\begin{gathered} p = x_4 x_5 x_6^{-1}, \quad q = x_5 x_6^{-1}, \quad r = x_6^{-1} \\ \gamma_1 = x_1 x_6^{-2}, \quad \gamma_2 = x_2 x_6^{-2}, \quad \gamma_3 = x_3 x_5 x_6^{-2} \end{gathered} \tag{4.32}$$

with the matrix

$$\tilde{\alpha} = \begin{Vmatrix} 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{Vmatrix}, \quad \det\tilde{\alpha} = 1 \tag{4.33}$$

and the time change $dt = x_6 d\tau$. Then system (4.10) takes the form

$$\begin{aligned} &\dot{x}_1 = x_2 - 2x_1x_2/c - x_3x_5^2, \quad \dot{x}_2 = -x_1 + 2x_2^2/c + x_3x_4x_5^2 \\ &\dot{x}_3 = x_1 - x_2x_4 - (c-1)x_3x_4 + x_3^2 - x_2x_3/c, \quad \dot{x}_4 = 1 - c + (1-c)x_4^2 + x_3x_4 \\ &\dot{x}_5 = [(c-1)x_4 - x_3 - x_2/c]x_5, \quad \dot{x}_6 = -x_2x_6/c \end{aligned} \tag{4.34}$$

where the dot denotes a derivative with respect to τ. As a result, the first integrals (1.3), (1.5) and (1.7) take the form

$$\begin{aligned} I_1 &= x_6^{-2}[c - 2x_1 + (x_4^2 + 1)x_5^2], \quad I_2 = x_6^{-3}x_5(x_1x_4 + x_2 + cx_3) \\ I_3 &= x_6^{-4}(x_1^2 + x_2^2 + x_3^2x_5^2), \quad I_4 = x_4x_5x_6^{-1} \\ \tilde{I}_4 &= x_6^{-4}[(2x_1 + x_4^2x_5^2 - x_5^2)^2 + 4(x_2 + x_4x_5^2)^2] \end{aligned} \tag{4.35}$$

Theorem 4.5 (Refs. 35,46, Section 5). *When $x_5 = x_6 = 0$, system (4.34) has the three families of fixed points*

$$\begin{aligned} \mathbf{A}: x_3 &= -i\frac{4c-3}{4c-2}, \quad x_4 = i\frac{2c-2}{2c-1} \\ \mathbf{B}: x_3 &= 0, \quad x_4 = -i \\ \mathbf{C}: x_3 &= ic(2-c), \quad x_4 = i(1-c) \end{aligned} \tag{4.36}$$

In all the families $x_1 = c/2$, and $x_2 = ic/2$. Families **A** *and* **B** *with $c \in (1/2, 1)$ correspond to two branches of F'_{10}, which are joined when $c = 3/4$. Family* **A** *with $c \in (0, 1/2)$ and family* **B** *with $c \in (1, 2)$ correspond to F'_8. Family* **C** *corresponds to F'_6.*

According to the discussion in Subsection 2.8, these families correspond to the following families of power-law asymptotic forms of the solutions of system (4.10) as $t \to 0$:

$$\begin{aligned} \mathbf{A}: p &= b_1t^{1-2c}, \quad q = b_2t^{1-2c}, \quad \gamma_3 = b_6t^{-2c} \\ \mathbf{B}: p &= b_1t^{2c-2}, \quad q = b_2t^{2c-2}, \quad \gamma_3 = b_6t^{2c-3} \\ \mathbf{C}: p &= b_1t^2, \quad q = b_2t^2, \quad \gamma_3 = b_6t \end{aligned} \tag{4.37}$$

In all the families $r = b_3t^{-1}$, $\gamma_1 = b_4t^{-2}$, and $\gamma_2 = b_5t^{-2}$, where the b_j are complex constants.

Family **A** is interesting when $c \in (0, 1)$, but $c \neq 1/2$. For the stationary points of family **A**, the vector of the eigenvalues of the matrix of system (4.34), linearized near these points, is

$$\Lambda = -i(1, 2, (2c+1)/2, (4c-3)/2, 1-c, 1/2) \tag{4.38}$$

Theorem 4.6 (Refs. 36,47, Section 6). *System (4.1) is locally integrable near the power-law asymptotic form (4.37) corresponding to* **A** *if $c \in (3/4, 1)$ or if an irrational $c \in (0, 3/4)$ is such that condition ω holds for the vector (4.38).*

Edneral wrote a program in the MATHEMATICA system for calculating a normal form[48] and used this program to calculate the normal form (4.7) near family **A** for several rational values of $c \in (0, 3/4)$ to the terms $g_{jQ}Z^Q$ of the high orders $\sum q_i \geq 6$. The calculated normal form was always linear. Therefore, it can be assumed that system (4.10) is always locally integrable near family **A** for $c \in (0, 3/4)$.

When $c = 3/4$, families **A** and **B** coinside, and, according to (4.38),

$$\Lambda = -i(1, 2, 5/4, 0, 1/4, 1/2) \tag{4.39}$$

The normal form was calculated to the 10th order. It is non-linear and does not satisfy Condition A in Subsection 4.2. This is consistent with the discussion in Subsection 2.6, where it was shown that the Kowalewski equations have a non-power-law asymptotic form of solutions with a two-parameter logarithm (the family G_4) when $c = 3/4$. Therefore, the system is locally non-integrable near the stationary point of the intersection of families $\mathbf{A} \cap \mathbf{B}$ when $c = 3/4$.

Family **B** is interesting when $c \in (1/2, 2]$. In this interval, the eigenvalue vector

$$\Lambda = -i(1, 2, (3-4c)/2, 2-c, (2c-1)/2, 1/2) \tag{4.40}$$

Theorem 4.7 (Refs. 36, 47, Section 7). *Near the power-law asymptotic form (4.37) corresponding to family* **B** *system (4.10) is locally integrable if $c \in (1/2, 3/4)$ or if an irrational $c \in (3/4, 2)$ is such that the vector Λ (4.40) satisfies Condition ω in* Subsection 4.2.

Normal forms were calculated for several rational values of $c \in (3/4, 2)$. For $c \neq 1, 3/2, 7/4$ they were also found to be linear.

When $c = 1$, according to (4.40),

$$\Lambda = -i(1, 2, -1/2, 1, 1/2, 1/2) \tag{4.41}$$

and the calculated normal form satisfies Condition A in Subsection 4.2, where α is a series in the positive powers $z_3^2 z_5^2$ and $\beta \equiv 0$.

Thus, near the asymptotic forms (4.37) corresponding to family **B**, system (4.10) is probably locally integrable.

Family **C** is interesting when $c \in (0, 2]$. In this interval,

$$\Lambda = -i(1, 2, -(2c+1)/2, c-2, 3/2, 1/2) \tag{4.42}$$

There are no analogues of Theorems 4.6 and 4.7 for it. Normal forms were calculated for several rational values of c. They turned out to linear for $c \neq 1/2, 2/3, 3/4, 1, 3/2, 7/4, 2$ and several other values. According to Theorem 4.2, the normalizing transformation converges in these cases. The linear normal form has five first integrals. Thus, in the vicinity of the power-law asymptotic forms (4.37) that correspond to family **C**, system (4.10) is locally integrable.

When $c = 1$, according to equality (4.42),

$$\Lambda = -i(1, 2, -3/2, -1, 3/2, 1/2)$$

and the calculated normal form satisfies Condition A in Subsection 4.2, where α is a series in $z_1 z_4$ and $z_3^2 z_5^2$, and $\beta \equiv 0$.

4.7. Periodic solutions

For periodic solutions there is a normal form theory that is similar to the theory for stationary solutions that was briefly discussed in Subsection 4.2. Let system (4.1) have the periodic solution

$$x_i = u_i(t), \quad i = 1, \dots, n \tag{4.43}$$

with period T: $U(t+T) = U(t)$, where $U = (u_1, \dots, u_n)$. We make the change $x_i = u_i(t) + \tilde{x}_i$ and write system (4.1) in the local coordinates $\tilde{x}_i$:

$$\tilde{x}_i' = \tilde{\varphi}_i(t, \tilde{X}), \quad i = 1, \dots, n \tag{4.44}$$

This system is periodic in t, and it has the zero solution $\tilde{X} = 0$ and the linear variational system

$$\tilde{X}'^* = A(t)\tilde{X}^* \tag{4.45}$$

where the $n \times n$ matrix $A(t) = (\partial\tilde{\varphi}_i/\partial\tilde{x}_i)$ when $\tilde{X} = 0$. Let $W(t)$ be the fundamental matrix of the solutions of the linear homogeneous system (4.45), and let $W^*(t+T) = MW^*(t)$, where M is the monodromy matrix. A periodic change

$$\tilde{X}^* = B(t)Y^*, \quad B(t+T) = B(t) \tag{4.46}$$

exists that transforms system (4.45) into the triangular system $Y^* = J(t)Y^*$, where the matrix $J(t)$ is the lower triangular with constant diagonal $\Lambda = (\lambda_1, \dots, \lambda_n)$. The entire system (4.44) takes the form

$$Y'^* = J(t)Y^* + \tilde{\tilde{\Phi}}^*(t, Y) \tag{4.47}$$

where $\tilde{\tilde{\Phi}}$ does not contain terms that are linear in Y or independent of Y. In addition, the matrices J and $\tilde{\tilde{\Phi}}$ are periodic in t. Let the formal change of coordinates that is periodic in t

$$Y = Z + \Xi(t, Z) \tag{4.48}$$

where $\Xi = (\xi_1, \dots, \xi_n)$ and $\xi_j(t, Z)$ are formal power series in Z without independent and linear terms, transform the system (4.47) into the system

$$\dot{Z}^* = J(t)Z^* + \Psi^*(t, Z) \tag{4.49}$$

We will write it in the form

$$\dot{z}_j = z_j g_j(t, Z) = z_j \sum g_{jpQ} Z^Q \exp(ip\kappa t) \text{ for } p \in \mathbb{Z}$$
$$Q \in \mathbb{N}_i, \quad i = 1, \ldots, n, \quad \kappa = 2\pi/T \tag{4.50}$$

System (4.49), (4.50) is called a *normal form* if a) the matrix $J(t)$ is a triangular matrix with a constant diagonal $\Lambda = (\lambda_1, \ldots, \lambda_n)$, and b) (4.50) contains only the resonant terms $g_{jPQ} Z^Q \exp(ip\kappa t)$, for which

$$ip\kappa + \langle Q, \Lambda \rangle = 0 \tag{4.51}$$

For any system (4.47) a formal change (4.48) exists that transforms it into the normal form (4.49), (4.50).

Condition A for a system that is periodic in t (Ref. 42, Section 11). A Poisson series

$$\alpha(t, Z) = \sum \alpha_{pQ} Z^Q \exp(ip\kappa t)$$

exists, such that in the normal form (4.50)

$$\psi_j = \lambda_j y_j + \alpha y_j \mathrm{Re}\lambda_j, \quad j = 1, \ldots, n$$

Theorem 4.8 (Ref. 42, Section 11). *If for an analytic system (4.47) the normal form (4.49), (4.50) satisfies Condition A and the vector (iκ, Λ) satisfies Condition ω in* Subsection 4.2, *the normalizing transformation (4.48) converges for sufficiently small* $|z_j|$.

For a periodic solution of the system of Euler–Poisson equations, we always have $\Lambda = (0, 0, 0, 0, \lambda_5, -\lambda_5)$. If $\lambda_5 \neq 0$ is real, the vector $(i\kappa, \Lambda)$ satisfies Condition ω. If, in addition, the periodic solution transforms into itself upon one of the automorphisms of the Euler–Poisson equations, the normal form for it is degenerate and satisfies Condition A. Therefore, according to Theorem 4.8, the normalizing transformation converges. At the same time, such a normal form has four formal first integrals, which are analytic owing to the convergence of the normalizing transformation. Thus, system (1.1) is locally integrable near such an unstable periodic solution.

The exact solutions of the Kowalewski Eq. (2.8) discussed in Section 3 correspond to the periodic solutions of the Euler–Poisson Eq. (1.1).[49] Among them there are solutions that are unstable in the linear approximations and are simultaneously symmetric.[50–52] System (1.1) is locally integrable near such periodic solutions.

5. Local non-integrability of the Euler–Poisson equations

There have been many different proofs of the absence of an additional first integral of the Euler–Poisson Eq. (1.1) in cases that differ from the classical cases of integrability.[53] The most advanced one is the proof[54] that system (1.1) does not have a meromorphic additional integral when $A = B$, $x_0 \neq 0$, $y_0 = 0$ if $A/C \neq 1, 2$. To prove this, on the invariant manifold $p = q = \gamma_2 = 0$, a one-parameter family of solutions expressed in terms of elliptic functions of the time t was examined.[54] The variational system was constructed for these solutions, its monodromy group was calculated, and it was shown that this group does not have properties that are characteristic for a linear integrable system. Thus, the absence of a local additional first integral near the solution chosen was proved.[54] Because of the complicated form of the solution chosen, this proof is also quite complicated.

Another method for proving the absence of a local additional first integral is proposed here. Its absence is proved (i) in the vicinity of a resonant stationary point and (ii) by calculating the coefficients of the normal form in the vicinity of the point indicated and verifying the necessary conditions for the existence of an additional formal integral. The advantage of this method is that it is algorithmic: all the calculations can be performed on a computer.

5.1. Structure of a normal form at resonance (Ref. 38, Section 6)

Consider system (4.14) in the sets D_4 and D_5 described in Subsection 4.4. Let

$$\lambda_5/\lambda_3 = \omega$$

According to Lemma 4.1, in the sets D_4 and D_5 all the eigenvalues λ_j are either pure imaginary or real, i.e., the ratio ω is a real number. To be specific, we will assume that $|\lambda_5|>|\lambda_3|$. Then $\omega>1$.

If ω is irrational, the normal form takes the form (4.28), (4.29). It also has four first integrals. However, in these cases Condition A in Subsection 4.2 does not hold in general. Consequently, now it cannot be asserted that system (4.14) has an additional local analytic first integral. The integrals (4.31) are only formal here.

Therefore, we will henceforth consider the case of a rational $\omega=\tilde{s}/\tilde{r}$, where $\tilde{s}>\tilde{r}\geq 1$ are mutually simple integers, i.e.,

$$\tilde{s}\lambda_3 = \tilde{r}\lambda_5 \tag{5.1}$$

We will investigate the structure of the normal form of system (4.14) at the resonance (5.1), following the previously described approach (Ref. 4, Chapter 5, Section 10). System (4.14) is system (4.2) with $n=6$. Let its normal form (4.6) be

$$\dot{z}_i = z_i g_i(Z), \quad i = 1, \dots, 6 \tag{5.2}$$

and let the vector Λ of the eigenvalues λ_j be

$$\Lambda = (0, 0, \lambda_3, -\lambda_3, \lambda_5, -\lambda_5), \quad \tilde{r}\lambda_5 = \tilde{s}\lambda_3, \quad 1\leq\tilde{r}<\tilde{s} \tag{5.3}$$

We set

$$\rho_1 = z_3z_4, \quad \rho_2 = z_5z_6, \quad w = z_3^{\tilde{s}}z_6^{\tilde{r}}, \quad \tilde{w} = z_4^{\tilde{s}}z_5^{\tilde{r}} \tag{5.4}$$

In this case $w\tilde{w}=\rho_1^{\tilde{s}}\rho_2^{\tilde{r}}$.

Lemma 5.1 ([38]). *At the resonance $\tilde{s}\lambda_3=\tilde{r}\lambda_5$ in the normal form (5.2)*

$$g_i = g_i(z_1, z_2, \rho_1, \rho_2, w, \tilde{w}) = g_{i0} + \sum_{k=1}^{\infty} f_{ik}w^k + \sum_{k=1}^{\infty} h_{ik}\tilde{w}^k, \quad i = 1, \dots, 6 \tag{5.5}$$

where z_1g_{10}, z_1f_{1k}, z_1h_{1k}, z_2g_{20}, z_2f_{2k}, z_2h_{2k}, g_{j0}, f_{3k}, ρ_1f_{4k}, ρ_2f_{5k}, f_{6k}, ρ_1h_{3k}, h_{4k}, h_{5k}, ρ_2h_{6k} *are power series in* z_1, z_2, ρ_1, ρ_2 ($j=3, 4, 5, 6$). *Here the series ρ_1h_{3k} and ρ_2f_{5k} begin with independent terms, and z_1g_{10}, z_2g_{20}, g_{j0} begin with linear terms.*

The proof is similar to the previously presented proof (Ref. 4, Chapter 5, Section 10, Lemma 10.2).

System (4.14) has the automorphisms (4.25) and

$$t, P, q, r, \Gamma, \gamma_2, \gamma_3 \to -t, P, q, -r, \Gamma, \gamma_2, -\gamma_3 \tag{5.6}$$

According to Property 1 in Subsection 4.2 and Lemma 4.2, the normal form (5.5) has the automorphisms (4.26) and

$$t, y_1, y_2, y_3, y_4, y_5, y_6 \to -t, y_1, y_2, -y_4, -y_3, -y_6, -y_5$$

where the y_j are replaced by z_j. For the resonant variables z_1, z_2 in (5.4), from these automorphisms we obtain the automorphisms

$$t, z_1, z_2, \rho_1, \rho_2, w, \tilde{w} \to -t, z_1, z_2, \rho_1, \rho_2, \tilde{w}, w \tag{5.7}$$

$$t, z_1, z_2, \rho_1, \rho_2, w, \tilde{w} \to -t, z_1, z_2, \rho_1, \rho_2, \tau\tilde{w}, \tau w \tag{5.8}$$

where

$$\tau = \begin{cases} 1, & \text{when} \quad \tilde{r}+\tilde{s}- \text{is even} \\ -1, & \text{when} \quad \tilde{r}+\tilde{s}- \text{is odd} \end{cases} \tag{5.9}$$

Consequently, when $\tilde{r}+\tilde{s}$ is even, automorphisms (5.7) and (5.8) are identical, whereas when it is odd, they are different and their product is the automorphism

$$t, z_1, z_2, \rho_1, \rho_2, w, \tilde{w} \to t, z_1, z_2, \rho_1, \rho_2, -w, -\tilde{w} \tag{5.10}$$

For the normal form (5.5), automorphism (4.26) and its corollary (5.7) give the equalities

$$g_i(z_1, z_2, \rho_1, \rho_2, w, \tilde{w}) = -g_i(z_1, z_2, \rho_1, \rho_2, \tilde{w}, w), \quad i = 1, 2$$

$$g_j(z_1, z_2, \rho_1, \rho_2, w, \tilde{w}) = -g_{j+1}(z_1, z_2, \rho_1, \rho_2, \tilde{w}, w), \quad j = 3, 5$$

or, more specifically,

$$g_{i0} \equiv 0, \quad f_{ik} = -h_{ik}, \quad i = 1, 2 \tag{5.11}$$

$$g_{j0} = -g_{j+1,0}, \quad f_{jk} = -h_{j+1,k}, \quad h_{jk} = -f_{j+1,k}, \quad j = 3, 5 \tag{5.12}$$

Finally, when $\tilde{r} + \tilde{s}$ is odd, it follows from automorphism (5.10) that the series g_i in the normal form (5.5) contains the variables w and $\tilde{w}$ only in even powers. Based on the normal form (5.5), (5.11), (5.12), we will construct a system of equations for the resonant variables after introducing the notation

$$\begin{aligned} &F_{1k} = f_{3k} - h_{3k} = f_{3k} + f_{4k}, \quad F_{2k} = f_{6k} - h_{6k} = f_{5k} + f_{6k} \\ &F_{3k} = \tilde{s}f_{3k} - \tilde{r}h_{5k} = \tilde{s}f_{3k} + \tilde{r}f_{6k}, \quad H_{3k} = \tilde{s}h_{3k} - \tilde{r}f_{5k} = \tilde{s}h_{3k} + \tilde{r}h_{6k} \\ &k = 1, 2, \ldots, \quad G_0 = \tilde{s}g_{30} - \tilde{r}g_{50} = \tilde{s}g_{30} + \tilde{r}g_{60} \end{aligned} \tag{5.13}$$

Taking into account that

$$H_{3k} = F_{3k} - \tilde{s}F_{1k} - \tilde{r}F_{2k}$$

we obtain the system

$$\begin{aligned} &\dot{z}_i = z_i \sum_{k=1}^{\infty} f_{ik}(w^k - \tilde{w}^k), \quad \dot{\rho}_i = \rho_i \sum_{k=1}^{\infty} F_{ik}(w^k - \tilde{w}^k); \quad i = 1, 2 \\ &\dot{w} = w\left[G_0 + \sum_{k=1}^{\infty} F_{3k}w^k + \sum_{k=1}^{\infty} H_{3k}\tilde{w}^k\right] \stackrel{\text{def}}{=} b_5(Z, \rho, w, \tilde{w}) \\ &\dot{\tilde{w}} = -\tilde{w}\left[G_0 + \sum_{k=1}^{\infty} F_{3k}\tilde{w}^k + \sum_{k=1}^{\infty} H_{3k}w^k\right] = -b_5(Z, \rho, \tilde{w}, w) \end{aligned} \tag{5.14}$$

where $\mathbf{z} = (z_1, z_2)$, $\rho = (\rho_1, \rho_2)$.

If $\tilde{r} + \tilde{s}$ is an odd number, the index k in the expansions (5.14) takes only the even values $2l$.

5.2. *First integrals (Ref. 38, Section 7)*

According to the previous discussion in Ref. 55, the expansion of the first integral of the normal form (5.2)

$$A = \sum a_Q Z^Q \tag{5.15}$$

contains only resonant terms, for which

$$\langle Q, \Lambda \rangle = 0 \tag{5.16}$$

Therefore, the first integral can be written in the form of a power series in the resonant variables

$$A = a_0 + \sum_{m=1}^{\infty} a_m w^m + \sum_{m=1}^{\infty} b_m \tilde{w}^m \tag{5.17}$$

where a_0, a_m, b_m are power series of $\mathbf{z}=(z_1, z_2)$, $\rho=(\rho_1, \rho_2)$. As a consequence of the automorphism (5.7), we have $a_m=b_m$, i.e., the integral can be written in the form

$$A = a_0 + \sum_{m=1}^{\infty} a_m(w^m + \tilde{w}^m) \tag{5.18}$$

If $\tilde{r}+\tilde{s}$ is an odd number, automorphism (5.10) dictates that this expansion contains only the even exponents $m=2n$, i.e.,

$$A = a_0 + \sum_{n=1}^{\infty} a_{2n}(w^{2n} + \tilde{w}^{2n}) \tag{5.19}$$

The derivative of the first integral (5.18) should vanish identically by virtue of system (5.14). Consequently,

$$\begin{aligned}
0 &\equiv \frac{\partial A}{\partial z_1}z_1g_1 + \frac{\partial A}{\partial z_2}z_2g_2 + \frac{\partial A}{\partial \rho_1}\rho_1(g_3+g_4) + \frac{\partial A}{\partial \rho_2}\rho_2(g_5+g_6) + \\
&+ \frac{\partial A}{\partial w}b_5(\mathbf{z}, \rho, w, \tilde{w}) - \frac{\partial A}{\partial \tilde{w}}b_5(\mathbf{z}, \rho, w, \tilde{w}) = \\
&= \sum_{k=1}^{\infty}\left(\frac{\partial a_0}{\partial z_1}z_1f_{1k} + \frac{\partial a_0}{\partial z_2}z_2f_{2k} + \frac{\partial a_0}{\partial \rho_1}\rho_1F_{1k} + \frac{\partial a_0}{\partial \rho_2}\rho_2F_{2k}\right)(w^k - \tilde{w}^k) + \\
&+ \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{\partial a_m}{\partial z_1}z_1f_{1k} + \frac{\partial a_m}{\partial z_2}z_2f_{2k} + \frac{\partial a_m}{\partial \rho_1}\rho_1F_{1k} + \frac{\partial a_m}{\partial \rho_2}\rho_2F_{2k}\right)(w^k - \tilde{w}^k)(w^m + \tilde{w}^m) + \\
&+ \sum_{m=1}^{\infty} ma_m\left[G_0(w^m - \tilde{w}^m) + \sum_{k=1}^{\infty}F_{3k}(w^{k+m} - \tilde{w}^{k+m}) + \sum_{k=1}^{\infty}H_{3k}(w^m\tilde{w}^k - w^k\tilde{w}^m)\right]
\end{aligned} \tag{5.20}$$

Let $\tilde{r}+\tilde{s}$ be an even number. Taking system (5.14) into account, we can write equality (5.20) for terms for which the total order with respect to z_3, z_4, z_5, z_6 is lower than $2(\tilde{r}+\tilde{s})$. For these terms $k=m=1$, and we obtain the equality

$$\frac{\partial a_0}{\partial z_1}z_1f_{11} + \frac{\partial a_0}{\partial z_2}z_2f_{21} + \frac{\partial a_0}{\partial \rho_1}\rho 1F_{11} + \frac{\partial a_0}{\partial \rho_2}\rho_2F_{21} + a_1G_0 = 0 \tag{5.21}$$

where in each component only terms of order lower than $\tilde{r}+\tilde{s}$ are retained. For an even value of $\tilde{r}+\tilde{s}$, the smallest possible value is equal to 4 (when $\tilde{r}=1$, $\tilde{s}=3$). Therefore, equality (5.21) should hold for a term that is independent of $\mathbf{z}$, ρ.

Let η_1, η_2, η_3, η_4 be terms that are independent of $\mathbf{z}$, ρ in the series z_1f_{11}, z_2f_{21}, ρ_1F_{11}, ρ_2F_{21} from system (5.14), respectively, and in expansion (5.18) let

$$a_0 = \text{const} + \alpha_1z_1 + \alpha_2z_2 + \alpha_3\rho_1 + \alpha_4\rho_2 + \ldots \tag{5.22}$$

Equality (5.21) for the independent term on the left-hand side gives the equation

$$\alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4 = 0$$

which has four linearly independent solutions $\alpha=(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ only if

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0 \tag{5.23}$$

This is a necessary condition for the existence of four functionally independent first integrals.

Now let $\tilde{r}+\tilde{s}$ be an odd number. Then, in equality (5.20) the indices k and m are even: $k=2l$ and $m=2n$. Therefore, taking system (5.14) into account and writing out equality (5.20) to terms with a total order with respect to z_3, z_4, z_5, z_6 that is lower than $4(\tilde{r}+\tilde{s})$, we obtain the equality

$$\frac{\partial a_0}{\partial z_1}z_1f_{12} + \frac{\partial a_0}{\partial z_2}z_2f_{22} + \frac{\partial a_0}{\partial \rho_1}\rho_1F_{12} + \frac{\partial a_0}{\partial \rho_2}\rho_2F_{22} + 2a_2G_0 = 0 \tag{5.24}$$

where only terms of order lower than $2(\tilde{r}+\tilde{s})$ are retained in each component. For an odd value of $\tilde{r}+\tilde{s}$, its smallest possible value is equal to $3(\tilde{r}=1, \tilde{s}=2)$. Therefore, equality (5.24) should hold for the term that is independent of **z**, ρ. Now let $\eta_1, \eta_2, \eta_3, \eta_4$ be terms that are independent of **z**, ρ in the series $z_1 f_{12}$, $z_2 f_{22}$, $\rho_1 F_{12}$, $\rho_2 F_{22}$, respectively, and let expansion (5.19) contain equality (5.22). Then, equalities (5.23) are also a necessary condition for the existence of four functionally independent first integrals.

5.3. 2:1 Resonance

We return to system (4.14). In the sets D_4 and D_5 the resonance $\lambda_5 = 2\lambda_3$ occurs on the two R_δ curves:

$$8y = 17(x-1) + 5\delta\sqrt{9x^2 - 34x + 41}, \quad \delta = \pm 1$$

(see the dashed curves in Fig. 5). On these curves, system (4.14) has the two automorphisms (4.25) and (5.6), and the results of Subsection 5.2 are applicable for odd $\tilde{r}+\tilde{s}=3$. The normal forms of system (4.14) were previously calculated[35,37] along the R_δ curves to sixth-order terms to obtain the values of η_3, η_4 as functions of $c \in (0, 2]$. The results are as follows. For $\delta = 1$ the equality $\eta_3 = \eta_4 = 0$ holds for only three values of c:

$$c_1 = 1, \quad c_2 = 1/2, \quad c_3 = 0.252778 \tag{5.25}$$

For $\delta = -1$ the equality $\eta_3 = \eta_4 = 0$ holds for only five values of c:

$$c_1 = 1, \quad c_2 = 1/2, \quad c_4 = 0.04522, \quad c_5 = 0.189372, \quad c_6 = 0.512902 \tag{5.26}$$

In addition, on the R_- curve the coefficients η_3 and η_4 go to infinity when

$$c_0 = 0.618034 \tag{5.27}$$

i.e., at the points (4.21). The intersection of the set (5.25) with the union of sets (5.26) and (5.27) consists of only two values c_1 and c_2, which correspond to two classical cases of global integrability. For other $c \in (0, 2]$ property (5.23) does not hold either on both R_δ curves or on one of them, and system (4.14) does not have an additional first integral.

When $y = \sigma p_0^2$ is real, the value of p_0 is real if $\sigma = \operatorname{sgn} y$; therefore, when y is real, the point (4.11) is real if it belongs to $\mathbf{S}_+$ for $y \geq 0$ and to $\mathbf{S}_-$ for $y \leq 0$.

System (1.1) has the linear automorphism

$$t, p, q, r, \gamma_1, \gamma_2, \gamma_3 \to -it, ip, iq, ir, -\gamma_1, -\gamma_2, -\gamma_3$$

Under this automorphism the families $\mathbf{S}_+$ and $\mathbf{S}_-$ transform one into another, and the normal forms for corresponding points of the families are similar in design and differ only with respect to the non-zero multipliers in front of the coefficients; therefore, the coefficients η_i in $\mathbf{S}_-$ vanish where and only where the same coefficients in $\mathbf{S}_+$ vanish. Thus, each point on the R_δ curves corresponds to a real point in $\mathbf{S}_+$, if $y \geq 0$, and to a real point in $\mathbf{S}_-$, if $y \leq 0$.

Thus, we have obtained the following theorem

Theorem 5.1. *For each* $c \in (0, 2]$, *except* $c = 1$ *and* $c = 1/2$, *a real stationary solution from the families* $\mathbf{S}_\sigma$ *exists, such that it belongs to the* R_δ *curves and system* (4.14) *is non-integrable in its vicinity.*

The values of η_1 and η_2 were also calculated as functions of C.[56] They vanish and go to infinity at the same values (5.25), (5.26) and (5.27) as do η_3 and η_4, i.e., condition (5.23) holds for the values (5.25) and (5.26).

Condition (5.23) is necessary, but is not sufficient for the existence of an additional integral. If it holds, the satisfaction of another condition that relates the coefficients of subsequent terms in the normal form is necessary for the existence of four integrals.[38] Direct calculations have established[56] that the other condition does not hold at c_3 on R_+ and at c_4, c_5 and c_6 on R_-, i.e., there is no additional local integral at these points. In fact, at these fixed points, there are additional families of periodic solutions of the set A (Ref. 41, Chapter 3, Section 3, Refs. 57–59, Part 2), which differ from the Lyapunov families and do not exist at other points on the R_δ curves.

The resonance $\lambda_5 = 3\lambda_3$ was also examined. In the sets D_4 and D_5 it occurs on two curves that exist for $c \in (0, 2]$ (see Ref. 38). Testing of property (5.23) on these curves gave the same rules: property (5.23) is valid simultaneously on both curves only for $c = 1$ and $c = 1/2$.[56] However, when this property holds, there are only additional families of periodic solutions, and there is no additional integral.

Remark. The absence of a formal additional integral near the stationary solution belonging to the intersection $\mathbf{S}_\sigma \cap R_\delta$ was, in fact, proved here. Hence it follows that there is no local additional analytic integral and no global additional analytic (and meromorphic) integral.

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